DIFFERENTIAL MANIFOLDS

Tammo tom Dieck

Mathematisches Institut

Georg-August-Universität

Göttingen

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Contents

Chapter 1

Manifolds

1.1 Differentiable Manifolds

A topological space X is n***-dimensional locally Euclidean*** if each $x \in X$ has an open neighbourhood U which is homeomorphic to an open subset V of $\mathbb{R}^n$. A homeomorphism $h\colon U \to V$ is a ***chart*** or ***local coordinate system*** of X about x with ***chart domain*** U. The inverse $h^{-1}\colon V \to U$ is a ***local parametrization*** of X about x. If $h(x) = 0$, we say that h and h^{-1} are ***centered*** at x. A set of charts is an ***atlas*** for X if their domains cover X. If X is n-dimensional locally Euclidean, we call n the ***dimension*** of X and write $\dim X = n$. The dimension is well-defined, by invariance of dimension.

An n***-dimensional manifold*** or just n***-manifold*** is an n-dimensional locally Euclidean Hausdorff space with countable basis for its topology. Hence manifolds are locally compact. A ***surface*** is a 2-manifold. A 0-manifold is a discrete space with at most a countably infinite number of points. The notation M^n is used to indicate that $n = \dim M$.

Suppose (U_1, h_1, V_1) and (U_2, h_2, V_2) are charts of an n-manifold. Then we have the associated ***coordinate change*** or ***transition function***

$$h_2 h_1^{-1}\colon h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2),$$

a homeomorphism between open subsets of Euclidean spaces.

Recall: A map $f\colon U \to V$ between open subsets of Euclidean spaces ($U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$) is a C^k-map if it is k-times continuously differentiable in the ordinary sense of analysis ($1 \le k \le \infty$). A continuous map is also called a C^0-map. A C^k-map $f\colon U \to V$ has a differential $Df(x)\colon \mathbb{R}^n \to \mathbb{R}^m$ at $x \in U$.

If the coordinate changes $h_2 h_1^{-1}$ and $h_1 h_2^{-1}$ are C^k-maps, we call the charts (U_1, h_1, V_1) and (U_2, h_2, V_2) C^k***-related*** ($1 \le k \le \infty$). An atlas is a C^k-atlas if any two of its charts are C^k-related. We call C^∞-maps ***smooth*** or just ***differentiable***; similarly, we talk about a smooth or differentiable atlas.

(1.1.1) Proposition. *Let $\mathcal{A}$ be a smooth atlas for M. The totality of charts which are smoothly related to all charts of $\mathcal{A}$ is a smooth atlas $D(\mathcal{A})$. If $\mathcal{A}$ and $\mathcal{B}$ are smooth atlasses, then $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas if and only if $D(\mathcal{A}) = D(\mathcal{B})$. The atlas $D(\mathcal{A})$ is the uniquely determined maximal smooth atlas which contains $\mathcal{A}$.* □

A ***differential structure*** on the n-manifold M is a maximal smooth atlas $\mathcal{D}$ for M. The pair $(M, \mathcal{D})$ is called a ***smooth manifold***. A maximal atlas serves just the purpose of this definition. Usually we work with a smaller atlas which then generates a unique differential structure. Usually we omit the differential structure from the notation; the charts of $\mathcal{D}$ are then called the charts of the differentiable manifold M.

Let M and N be smooth manifolds. A map $f\colon M \to N$ is ***smooth*** at $x \in M$ if f is continuous at x and if for charts (U, h, U') about x and (V, k, V') about $f(x)$ the composition kfh^{-1} is differentiable at $h(x)$. We call kfh^{-1} the ***expression of f in local coordinates***. The map f is smooth if it is differentiable at each point. The composition of smooth maps is smooth. Thus we have the category of smooth manifolds and smooth maps. A ***diffeomorphism*** is a smooth map which has a smooth inverse. Manifolds M and N are ***diffeomorphic*** if there exists a diffeomorphism $f\colon M \to N$.

Let $U \subset \mathbb{R}^n$ be open. Then (U, id, U) is a chart for U and already an atlas. This atlas makes U into a smooth manifold. When we talk about U as a smooth manifold, we think of this structure.

Let U be open in a smooth manifold M. The totality of charts of M with domain in U form a smooth atlas for U. We call such manifolds U ***open submanifolds*** of M. The reader should now verify two things: (1) The inclusion $U \subset M$ is then a smooth map. (2) A chart (U, h, V) of a smooth n-manifold M is a diffeomorphism of the open submanifold U onto the open submanifold V of $\mathbb{R}^n$.

Smooth manifolds M and N have a ***product*** in the category of smooth manifolds. The charts of the form $(U \times V, f \times g, U' \times V')$ for charts (U, f, U') of M and (V, g, V') of N define a smooth structure on $M \times N$. The projections onto the factors are smooth. The canonical isomorphisms $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ are diffeomorphisms.

A subset N of an n-manifold M is a ***k-dimensional submanifold*** of M if the following holds: For each $x \in N$ there exists a chart $h\colon U \to U'$ of M about x such that $h(U \cap N) = U' \cap (\mathbb{R}^k \times 0)$. A chart with this property is called ***adapted*** to N. The difference $n - k$ is the ***codimension*** of N in M. (The subspace $\mathbb{R}^k \times 0$ of $\mathbb{R}^n$ may be replaced by any k-dimensional linear or affine subspace if this is convenient.) If we identify $\mathbb{R}^k \times 0 = \mathbb{R}^k$, then $(U \cap N, h, U' \cap \mathbb{R}^k)$ is a chart of N. If M is smooth, we call N a ***smooth submanifold*** of M if there exists about each point an adapted chart from the differential structure of M. The totality of charts $(U \cap N, h, U' \cap \mathbb{R}^k)$ which arise from adapted smooth charts of M is then a smooth atlas for N. In this way, a differentiable submanifold becomes a smooth manifold, and the inclusion $N \subset M$ is a smooth map. A smooth map $f\colon N \to M$ is a ***smooth embedding*** if $f(N) \subset M$ is a smooth submanifold and $f\colon N \to f(N)$ a diffeomorphism.

(1.1.2) Spheres. The spheres are manifolds which need an atlas with at least two charts. We have the atlas with two charts (U_N, φ_N) and (U_S, φ_S) coming from the stereographic projection (**??**). The coordinate transformation is $\varphi_S \circ \varphi_N^{-1}(y) = \|y\|^{-2}y$. The differential of the coordinate transformation at x is $\xi \mapsto (\|x\|^2\xi - 2\langle x, \xi \rangle x) \cdot \|x\|^{-4}$. For $\|x\| = 1$ we obtain the reflection $\xi \mapsto \xi - 2\langle x, \xi \rangle x$ in a hyperplane. ◇

(1.1.3) Projective Spaces. We construct charts for the projective space $\mathbb{R}P^n$. The subset $U_i = \{[x_0, \dots, x_n] \mid x_i \neq 0\}$ is open. The assignment

$$\varphi_i\colon U_i \to \mathbb{R}^n, \quad [x_0, \dots, x_n] \mapsto x_i^{-1}(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

is a homeomorphism. These charts are smoothly related.

Charts for $\mathbb{C}P^n$ can be defined by the same formulas. Note that $\mathbb{C}P^n$ has dimension $2n$ as a smooth manifold. (It is n-dimensional when viewed as a so-called complex manifold.) ◇

(1.1.4) Proposition. *Let M be an n-manifold and $\mathcal{U} = (U_j \mid j \in J)$ an open covering of M. Then there exist charts $(V_k, h_k, B_k \mid k \in \mathbb{N})$ of M with the following properties:*

(1) *Each V_k is contained in some member of $\mathcal{U}$.*
(2) *$B_k = U_3(0)$.*
(3) *The family $(V_k \mid k \in \mathbb{N})$ is a locally finite covering of M.*

In particular, each open cover has a locally finite refinement, i.e., manifolds are paracompact. If M is smooth, there exists a smooth partition of unity $(\sigma_k \mid k \in \mathbb{N})$ subordinate to (V_k). There also exists a smooth partition of unity $(\alpha_j \mid j \in J)$ such that the support of α_j is contained in U_j and at most a countable number of the α_j are non-zero.

Proof. The space M is a locally compact Hausdorff space with a countable basis. Therefore there exists an exhaustion

$$M_0 \subset M_1 \subset M_2 \subset \ldots \subset M = \textstyle\bigcup_{i=1}^{\infty} M_i$$

by open sets M_i such that $\overline{M}_i$ is compact and contained in M_{i+1} (**??**). Hence $K_i = \overline{M}_{i+1} \smallsetminus M_i$ is compact. For each i we can find a finite number of charts (V_ν, h_ν, B_ν), $B_\nu = U_3(0)$, such that $V_\nu \subset U_j$ for some j and such that the $h_\nu^{-1}U_1(0)$ cover K_i and such that $V_\nu \subset M_{i+2} \smallsetminus \overline{M}_{i-1}$ $(M_{-1} = \emptyset)$. Then the V_ν form a locally finite, countable covering of M, now denoted $(V_k, h_k, B_k \mid k \in \mathbb{N})$.

The function $\lambda\colon \mathbb{R} \to \mathbb{R}$, $\lambda(t) = 0$ for $t \le 0$, $\lambda(t) = \exp(-1/t)$ for $t > 0$, is a C^∞-function. For $\varepsilon > 0$, the function $\varphi_\varepsilon(t) = \lambda(t)(\lambda(t) + \lambda(\varepsilon - t))^{-1}$ is C^∞ and satisfies $0 \le \varphi_\varepsilon \le 1$, $\varphi_\varepsilon(t) = 0 \Leftrightarrow t \le 0$, $\varphi_\varepsilon(t) = 1 \Leftrightarrow t \ge \varepsilon$. Finally, $\psi\colon \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \varphi_\varepsilon(\|x\| - r)$ is a C^∞-map which satisfies $0 \le \psi(x) \le 1$, $\psi(x) = 1 \Leftrightarrow x \in U_r(0)$, $\psi(x) = 0 \Leftrightarrow \|x\| \ge r + \varepsilon$.

We use these functions ψ for $r = 1$ and $\varepsilon = 1$ and define ψ_i by $\psi \circ h_i$ on V_i and as zero on the complement. Then the $\sigma_k = s^{-1}\psi_k$ with $s = \sum_{j=1}^\infty \psi_j$ yield a smooth, locally finite partition of unity subordinate to $(V_k \mid k \in \mathbb{N})$.

The last statement follows from **??**. □

(1.1.5) Example. Let C_0 and C_1 be closed disjoint subsets of the smooth manifold M. Then there exists a smooth function $\varphi\colon M \to [0,1]$ such that $\varphi(C_j) \subset \{j\}$; apply the previous proposition to the covering by the $U_j = M \smallsetminus C_j$. ◇

(1.1.6) Example. Let A be a closed subset of the smooth manifold M and U an open neighbourhood of A in M. Let $f\colon U \to [0,1]$ be smooth. Then there exists a smooth function $F\colon M \to [0,1]$ such that $F|A = f|A$. For the proof choose a partition of unity (φ_0, φ_1) subordinate to $(U, M \smallsetminus A)$. Then set $F(x) = \varphi_0(x)f(x)$ for $x \in U$ and $F(x) = 0$ otherwise. ◇

(1.1.7) Proposition. *Let M be a submanifold of N. A smooth function $f\colon M \to \mathbb{R}$ has a smooth extension $F\colon N \to \mathbb{R}$.*

Proof. From the definition of a submanifold we obtain for each $p \in M$ an open neighbourhood U of p in N and a smooth retraction $r\colon U \to U \cap M$. Hence we can find an open covering $(U_j \mid j \in J)$ of M in N and smooth extensions $f_j\colon U_j \to \mathbb{R}$ of $f|U_j \cap M$. Let $(\alpha_j \mid j \in J)$ be a subordinate smooth partition of unity and set $F(x) = \sum_{j\in J} \alpha_j(x) f_j(x)$, where a summand is defined to be zero if $f_j(x)$ is not defined. □

(1.1.8) Proposition. *Let M be a smooth manifold. There exists a smooth proper function $f\colon M \to \mathbb{R}$.*

Proof. A function is proper if the pre-image of a compact set is compact (**??**). We choose a countable partition of unity $(\tau_k \mid k \in \mathbb{N})$ such that the functions τ_k have compact support. Then we set $f = \sum_{k=1}^\infty k \cdot \tau_k\colon M \to \mathbb{R}$. If $x \notin \bigcup_{j=1}^n \operatorname{Supp}(\tau_j)$, then $1 = \sum_{j\ge 1} \tau_j(x) = \sum_{j>n} \tau_j(x)$ and therefore $f(x) = \sum_{j>n} j\tau_j(x) > n$. Hence $f^{-1}[-n,n]$ is contained in $\bigcup_{j=1}^n \operatorname{Supp}(\tau_j)$ and therefore compact. □

In working with submanifolds we often use, without further notice, the following facts. Let M be a smooth manifold and $A \subset M$. Then A is a submanifold if and only if each $a \in A$ has an open neighbourhood U such that $A \cap U$ is a submanifold of U. (Being a submanifold is a local property.) Let

$f\colon N_1 \to N_2$ be a diffeomorphism. Then $M_1 \subset N_1$ is a submanifold if and only if $f(M_1) = M_2 \subset N_2$ is a submanifold. (Being a submanifold is invariant under diffeomorphisms.)

The definition of a manifold via charts admits a different interpretation. The manifold is obtained as an identification space from the domains of definition of the local parametrizations. The identification is given by the coordinate changes. It turns out that the coordinate changes are the basic structural data. In particular, they determine the topology. We have formalized the identification process in ??; we use the notation of that section.

Suppose in addition to the hypothesis of**??** that the U_i are smooth n-manifolds and the g_i^j are diffeomorphisms. Then X is n-dimensional locally Euclidean. If X is a Hausdorff space and has a countable basis, then X carries a unique structure of a smooth n-manifold such that the h_i are smooth embeddings onto open submanifolds. The simplest situation arises, when we have two manifolds U_1 and U_2 with open subsets $V_j \subset U_j$ and a gluing diffeomorphism $\varphi\colon V_1 \to V_2$. Then $X = U_1 \cup_\varphi U_2$ is the manifold obtained from $U_1 + U_2$ by identifying $v \in V_1$ with $\varphi(v) \in V_2$.

(1.1.9) Example. Let $U_1 = U_2 = \mathbb{R}^n$ and $V_1 = V_2 = \mathbb{R}^n \setminus 0$. Let $\varphi = \mathrm{id}$. Then the graph of φ in $\mathbb{R}^n \times \mathbb{R}^n$ is not closed. The resulting locally Euclidean space is not Hausdorff. If we use $\varphi(x) = x \cdot \|x\|^{-2}$, then the result is a compact n-manifold. It is diffeomorphic to S^n. ◇

Important objects are the group objects in mathematics are the smooth category. A ***Lie group*** consists of a smooth manifold G and a group structure on G such that the group multiplication and the passage to the inverse are smooth maps. The fundamental examples are the classical matrix groups. A basic result in this context says that a closed subgroup of a Lie group is a submanifold and with the induced structure a Lie group [**?**] [**?**].

Problems

1. Let E be an n-dimensional real vector space $0 < r < n$. We define charts for the Grassmann manifold $G_r(E)$ of r-dimensional subspaces of E. Let K be a subspace of codimension r in E. Consider the set of complements in K

$$U(K) = \{F \in G_r(E) \mid F \oplus K = E.\}$$

The sets are the chart domains. Let $P(K) = \{p \in \mathrm{Hom}(E,E) \mid p^2 = p, p(E) = K\}$ be the set of projections with image K. Then $P(K) \to U(K)$, $p \mapsto \mathrm{Ker}(p)$ is a bijection. The set $P(K)$ is an affine space for the vector space $\mathrm{Hom}(E/K, K)$. Let $j\colon K \subset E$ and $q\colon E \to E/K)$ the quotient map. Then

$$\mathrm{Hom}(E/K, K) \times P(K) \to P(K), \quad (\varphi, p) \mapsto p + j\varphi q$$

is a transitive free action. We choose a base point $p_0 \in P(K)$ in this affine space and obtain a bijection

$$U(K) \leftarrow P(K) \rightarrow \mathrm{Hom}(E/K, K), \quad \mathrm{Ker}(p) \hookleftarrow p \mapsto p - p_0.$$

The bijections are the charts for a smooth structure.
2. $\{(x,y,z) \in \mathbb{R}^3 \mid z^2x^3 + 3zx^2 + 3x - zy^2 - 2y = 1\}$ is a smooth submanifold of $\mathbb{R}^3$ diffeomorphic to $\mathbb{R}^2$. If one considers the set of solution $(x,y,z) \in \mathbb{C}^2$, then one obtains a smooth complex submanifold of $\mathbb{C}^3$ which is contractible but not homeomorphic to $\mathbb{C}^2$ [**?**].

1.2 Tangent Spaces and Differentials

We associate to each point p of a smooth m-manifold M an m-dimensional real vector space $T_p(M)$, the ***tangent space*** of M at the point p, and to each smooth map $f\colon M \to N$ a linear map $T_pf\colon T_p(M) \to T_{f(p)}(N)$, the ***differential*** of f at p, such that the functor properties hold (***chain rule***)

$$T_p(gf) = T_{f(p)}g \circ T_pf, \quad T_p(\mathrm{id}) = \mathrm{id}\,.$$

The elements of $T_p(M)$ are the ***tangent vectors*** of M at p.

Since there exist many different constructions of tangent spaces, we define them by a universal property.

A ***tangent space*** of the m-dimensional smooth manifold M at p consists of an m-dimensional vector space $T_p(M)$ together with an isomorphism $i_k\colon T_pM \to \mathbb{R}^m$ for each chart $k = (U, \varphi, U')$ about p such that for any two such charts k and $l = (V, \psi, V')$ the isomorphism $i_l^{-1}i_k$ is the differential of the coordinate change $\psi\varphi^{-1}$ at $\varphi(p)$. If (T'_pM, i'_k) is another tangent space, then $\iota_p = i_k^{-1} \circ i'_k\colon T'_pM \to T_pM$ is independent of the choice of k. Thus a tangent space is determined, up to unique isomorphism, by the universal property. If we fix a chart k, an *arbitrary* m-dimensional vector space T_pM, and an isomorphism $i_k\colon T_pM \to \mathbb{R}^m$, then there exists a unique tangent space with underlying vector space T_pM and isomorphism i_k; this follows from the chain rule of calculus. Often we talk about the tangent space T_pM and understand a suitable isomorphism $i_k\colon T_pM \to \mathbb{R}^m$ as structure datum.

Let $f\colon M^m \to N^n$ be a smooth map. Choose charts $k = (U, \varphi, U')$ about $p \in M$ and $l = (V, \psi, V')$ about $f(p) \in N$. There exists a unique linear map T_pf which makes the diagram

$$\begin{array}{ccc} T_pM & \xrightarrow{T_pf} & T_{f(p)}N \\ \uparrow{\scriptstyle i_k} & & \uparrow{\scriptstyle i_l} \\ \mathbb{R}^m & \xrightarrow{D(\psi f \varphi^{-1})} & \mathbb{R}^n \end{array}$$

commutative; the morphism at the bottom is the differential of $\psi f \varphi^{-1}$ at $\varphi(p)$. Again by the chain rule, $T_p f$ is independent of the choice of k and l. Differentials, defined in this manner, satisfy the chain rule. This definition is also compatible with the universal maps ι_p for different choices of tangent spaces $T_p f \circ \iota_p = \iota_{f(p)} \circ T_p' f$.

In abstract terms: Make a choice of $T_p(M)$ for each pair $p \in M$. Then the $T_p M$ and the $T_p f$ constitute a functor from the category of pointed smooth manifolds and pointed smooth maps to the category of real vector spaces. Different choices of tangent spaces yield isomorphic functors.

The purpose of tangent spaces is to allow the definition of differentials. The actual vector spaces are adapted to the situation at hand and can serve other geometric purposes (e.g. they can consist of geometric tangent vectors).

(1.2.1) Examples. (1) If $V \subset \mathbb{R}^n$ is an open subset, we set $T_p V = \mathbb{R}^n$ and $i_k = \mathrm{id}$ for $k = (V, \mathrm{id}, V)$. In this way, we identify $T_p V$ with $\mathbb{R}^n$. Under this identification, $T_p f$ for a smooth map $f \colon U \to V$ between open subsets of Euclidean spaces becomes the ordinary differential $Df(p)$ of calculus.

(2) Let $j \colon W \subset M$ be the inclusion of an open subset of a smooth manifold M and let $x \in W$. Fix a chart $k = (U, h, V)$ of M about x such that $U \subset W$. Then k is also a chart for the open submanifold U. Given $T_x M$ and i_k, we set $T_x W = T_x M$ and use the same i_k for W. Then $T_x j \colon T_x W \to T_x M$ is the identity.

(3) Let $k = (U, h, V)$ be a chart of M about p. We use the conventions of (1) and (2). Then $T_p h \colon T_p U \to T_{h(p)} V = \mathbb{R}^m$ is i_k.

(4) Let $i \colon M \to N$ be the inclusion of a submanifold. Then $T_p i$ is injective, because in local coordinates with an adapted chart, i is the inclusion of a subspace (restricted to open subsets), and the differential of i is this inclusion. Given $T_p N$, the image of $T_p i$ is independent of the choice of $T_p M$. We often therefore take this image as a model for $T_p M$. More precisely: If $K = (U, \Phi, V)$, $V \subset \mathbb{R}^n$ is an adapted chart and $k = (U \cap M, \varphi, W)$, $W \subset \mathbb{R}^m \cong \mathbb{R}^m \times 0 \subset \mathbb{R}^n$ its restriction, then we use

$$\begin{array}{ccc} T_p N & \xrightarrow{i_K} & \mathbb{R}^n \\ \uparrow \cup & & \uparrow \\ T_p M & \xrightarrow{i_k} & \mathbb{R}^m \end{array}$$

as structure data, and $T_p i$ becomes the inclusion $T_p M \subset T_p N$.

In particular, if M is a submanifold of $\mathbb{R}^n$, then $T_p M$ is identified with a subspace of $\mathbb{R}^n$. This subspace has the following interpretation in terms of tangents. Let $\alpha \colon]-\varepsilon, \varepsilon[\to M$ be a smooth curve with $\alpha(0) = p$. Then the derivative $\alpha'(0) = \frac{d\alpha}{dt}(0) \in \mathbb{R}^n$ is contained in this subspace $T_p M$, and $T_p M$ is the totality of such "velocity vectors" of curves α.

Suppose we have a commutative diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{f_0} & N_0 \\ \downarrow \cap & & \downarrow \cap \\ M & \xrightarrow{f} & N \end{array}$$

of smooth maps and submanifolds. The chain rule tells us that under the identifications $T_pM_0 \subset T_pM, T_qN_0 \subset T_qN$ the differential $T_p(f_0)$ coincides with the restriction of $T_p(f)$ to $T_p(M_0)$. ◇

We call a smooth map f an ***immersion*** if each differential T_xf is injective and a ***submersion*** if each differential T_xf is surjective. The point $x \in M$ is a ***regular*** point of f if T_xf is surjective. A point $y \in N$ is a ***regular value*** of f if each $x \in f^{-1}(y)$ is a regular point, and otherwise a ***singular value***. If $f^{-1}(y) = \emptyset$, then y is also called a regular value.

(1.2.2) Rank Theorem. *Let $f\colon M \to N$ be a smooth map from an m-manifold into an n-manifold.*

(1) *If T_af is bijective, then there exist open neighbourhoods U of a and V of $f(a)$, such that f induces a diffeomorphism $f\colon U \to V$.*

(2) *If T_af is injective, then there exist open neighbourhoods U of a, V of $f(a)$, W of $0 \in \mathbb{R}^{n-m}$ and a diffeomorphism $F\colon U \times W \to V$ such that $F(x,0) = f(x)$ for $x \in U$.*

(3) *If T_af is surjective, then there exist open neighbourhoods U of a, V of $f(a)$, W of $0 \in \mathbb{R}^{m-n}$ and a diffeomorphism $F\colon U \to V \times W$ such that $\mathrm{pr}_V F(x) = f(x)$ for $x \in U$ with the projection $\mathrm{pr}_V\colon V \times W \to V$.*

(4) *Suppose T_xf has rank r for all $x \in M$. Then for each $a \in M$ there exist open neighbourhoods U of a, V of $f(a)$ and diffeomorphisms $\varphi\colon U \to U'$, $\psi\colon V \to V'$ onto open sets $U' \subset \mathbb{R}^m$, $V' \subset \mathbb{R}^n$ such that $f(U) \subset V$ and $\psi f \varphi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0)$ for all $(x_1, \ldots, x_m) \in U'$.*

Proof. The assertions are of a local nature. Therefore we can, via local charts, reduce to the case that M and N are open subsets of Euclidean spaces. Then these assertions are known from calculus. □

(1.2.3) Proposition. *Let y be a regular value of the smooth map $f\colon M \to N$. Then $P = f^{-1}(y)$ is a smooth submanifold of M. For each $x \in P$, we can identify T_xP with the kernel of T_xf.*

Proof. Let $x \in P$. The rank theorem 1.2.2 says that f is in suitable local coordinates about x and $f(x)$ a surjective linear map; hence P is locally a submanifold.

The differential of a constant map is zero. Hence T_xP is contained in the kernel of T_xf. For reasons of dimension, the spaces coincide. □

(1.2.4) Example. The differentials of the projections onto the factors yield an isomorphism $T_{(x,y)}(M \times N) \cong T_x(M) \times T_y(N)$ which we use as an identification. With these identifications, $T_{(x,y)}(f \times g) = T_x f \times T_y g$ for smooth maps f and g. Let $h\colon M \times N \to P$ be a smooth map. Then $T_{(x,y)}h$, being a linear map, is determined by the restrictions to $T_x M$ and to $T_y N$, hence can be computed from the differentials of the partial maps $h_1\colon a \mapsto h(a,y)$ and $h_2\colon b \mapsto h(x,b)$ via $T_{(x,y)}h(u,v) = T_x h_1(u) + T_y h_2(v)$. ◇

(1.2.5) Proposition. *Suppose $f\colon M \to N$ is an immersion which induces a homeomorphism $M \to f(M)$. Then f is a smooth embedding.*

Proof. We first show that $f(M)$ is a smooth submanifold of N of the same dimension as M. It suffices to verify this locally.

Choose U, V, W and F according to 1.2.2. Since U is open and $M \to f(M)$ a homeomorphism, the set $f(U)$ is open in $f(M)$. Therefore $f(U) = f(M) \cap P$, with some open set $P \subset N$. The set $R = V \cap P$ is an open neighbourhood of b in N, and $R \cap f(M) = f(U)$ holds by construction. It suffices to show that $f(U)$ is a submanifold of R. We set $Q = F^{-1}R$, and have a diffeomorphism $F\colon Q \to R$ which maps $U \times 0$ bijectively onto $f(U)$. Since $U \times 0$ is a submanifold of $U \times W$, we see that $f(U)$ is a submanifold.

By assumption, $f\colon M \to f(M)$ has a continuous inverse. This inverse is smooth, since $f\colon M \to f(M)$ has an injective differential, hence bijective for dimensional reasons, and is therefore a local diffeomorphism. □

(1.2.6) Proposition. *Let $f\colon M \to N$ be a surjective submersion and $g\colon N \to P$ a set map between smooth manifolds. If gf is smooth, then g is smooth.*

Proof. Let $f(x) = y$. By the rank theorem, there exist chart domains U about x and V about y such that $f(U) = V$ and $f\colon U \to V$ has, in suitable local coordinates, the form $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_n)$. Hence there exists a smooth map $s\colon V \to U$ such that $fs(z) = z$ for all $z \in V$. Then $g(z) = gfs(z)$, and gfs is smooth. (The map s is called a ***local section*** of f about y.) □

We now give an intrinsic and canonical construction of tangent vectors as directional derivatives of smooth functions. Let $\mathcal{E}_x(M)$ be the $\mathbb{R}$-algebra of ***germs*** of smooth functions $(M,x) \to \mathbb{R}$. Elements in $\mathcal{E}_x(M)$ are represented by smooth functions defined in a neighbourhood of x and will be denoted by their representatives; two such define the same element of $\mathcal{E}_x(M)$ if they coincide in some neighbourhood of x. Addition and multiplication is defined pointwise on representatives. A ***derivation*** of $\mathcal{E}_x(M)$ is a linear map $D\colon \mathcal{E}_x(M) \to \mathbb{R}$ such that $D(f \cdot g) = D(f) \cdot g(x) + f(x) \cdot D(g)$. Let $\mathcal{T}_x(M)$ be the vector space of derivations of $\mathcal{E}_x(M)$. A smooth map $f\colon M \to N$ induces a homomorphism of algebras

$$\mathcal{E}_x f\colon \mathcal{E}_{f(x)}N \to \mathcal{E}_x M, \quad \varphi \mapsto \varphi \circ f$$

and a linear map

$$\mathcal{T}_x(f)\colon \mathcal{T}_x(M) \to \mathcal{T}_{f(p)}(N), \quad D \mapsto D \circ \mathcal{E}_x f.$$

This construction is functorial. The next lemma from calculus is used to show that these data can be used to exhibit the $\mathcal{T}_x M$ as tangent spaces.

(1.2.7) Lemma. *The vector space $\mathcal{T}_p\mathbb{R}^n$ has a basis consisting of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. Here we view $\sum_i a_i \frac{\partial}{\partial x_i}$ as the derivation $f \mapsto \sum_i a_i \frac{\partial f}{\partial x_i}(p)$ which arises from the standard coordinates $x_1, \ldots, x_n$ of $\mathbb{R}^n$. If we identify $\mathcal{T}_p(\mathbb{R}^n)$ via this basis with $\mathbb{R}^n$, then $\mathcal{T}_p(\varphi)$, for a smooth map φ between open sets of Euclidean spaces, is given by the Jacobi-matrix of $D\varphi(p)$.* □

From this lemma it follows that the $i_k = \mathcal{T}_x\varphi\colon \mathcal{T}_x(M) \to \mathcal{T}_{\varphi(x)}(V) = \mathbb{R}^m$ for charts $k = (U, \varphi, V)$ form a tangent space. With this model of tangent spaces, we have for each $X_p \in T_p(M)$ and each smooth function $f\colon M \to \mathbb{R}$ the derivative $X_p f$ of f in direction X_p.

Problems

1. An injective immersion of a compact manifold is a smooth embedding.
2. Let $f\colon M \to N$ be a smooth map which induces a homeomorphism $M \to f(M)$. If the differential of f has constant rank, then f is a smooth embedding. By the rank theorem, f has to be an immersion, since f is injective.
3. Let M be a smooth m-manifold and $N \subset M$. The following assertions are equivalent: (1) N is a k-dimensional smooth submanifold of M. (2) For each $a \in N$ there exist an open neighbourhood U of a in M and a smooth map $f\colon U \to \mathbb{R}^{m-k}$ such that the differential $Df(u)$ has rank $m-k$ for all $u \in U$ and such that $N \cap U = f^{-1}(0)$. (Submanifolds are locally solution sets of "regular" equations.)
4. The graph of a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth submanifold of $\mathbb{R}^{n+1}$.
5. Let Y be a smooth submanifold of Z and $X \subset Y$. Then X is a smooth submanifold of Y if and only if it is a smooth submanifold of Z. If X is a smooth submanifold, then there exists about each point $x \in X$ a chart (U, φ, V) of Z such that $\varphi(U \cap X)$ as well as $\varphi(U \cap Y)$ are intersections of V with linear subspaces. (Charts ***adapted*** to $X \subset Y \subset Z$. Similarly for inclusions $X_1 \subset X_2 \subset \ldots \subset X_k$.)
6. The defining map $\mathbb{R}^{n+1} \smallsetminus 0 \to \mathbb{R}P^n$ is a submersion. Its restriction to S^n is a submersion and an immersion (a 2-fold regular covering).

1.3 The theorem of Sard

It is an important fact of analysis that most values are regular. A set $A \subset N$ in the n-manifold N is said to have (Lebesgue) ***measure zero*** if for each chart

(U, h, V) of N the subset $h(U \cap A)$ has measure zero in $\mathbb{R}^n$. A subset of $\mathbb{R}^n$ has measure zero if it can be covered by a countable number of cubes with arbitrarily small total volume. We use the fact that a diffeomorphism (in fact a C^1-map) sends sets of measure zero to sets of measure zero. An open (non-empty) subset of $\mathbb{R}^n$ does not have measure zero. The next theorem is a basic result for differential topology. (Proofs [**?**], [**?**], [**?**].)

(1.3.1) Theorem (Sard). *The set of singular values of a smooth map has measure zero, and the set of regular values is dense.*

Proof. □

1.4 Examples

(1.4.1) Example. $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $(x_0, \dots, x_n) \mapsto \sum x_i^2 = \|x\|^2$ has, away from the origin, a non-zero differential. The sphere

$$S^n(c) = f^{-1}(c^2) = \{x \in \mathbb{R}^{n+1} \mid c = \|x\|\}$$

of radius $c > 0$ is therefore a smooth submanifold of $\mathbb{R}^{n+1}$. From 1.2.3 we obtain $T_x S^n(c) = \{v \in \mathbb{R}^{n+1} \mid x \perp v\}$. ◇

(1.4.2) Example. Let $M(m, n)$ be the vector space of real (m, n)-matrices and $M(m, n; k)$ for $0 \le k \le \min(m, n)$ the subset of matrices of rank k. Then $M(m, n; k)$ is a smooth submanifold of $M(m, n)$ of dimension $k(m + n - k)$. For the proof, write a matrix $X \in M(m, n; k)$ in block form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with a (k, k)-matrix A. The subset $U = \{X \in M(m, n) \mid \det(A) \neq 0\}$ is open. The relation

$$\begin{pmatrix} E & 0 \\ -CA^{-1} & E \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

shows that $X \in U$ has rank k if and only if $D = CA^{-1}B$. The map

$$\varphi\colon U \to M(m - k, n - k), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto D - CA^{-1}B$$

satisfies $\varphi^{-1}(0) = U \cap M(m, n; k)$, and its differential has rank $(m - k)(n - k)$ everywhere, as can be seen by varying D alone. This shows that $U \cap M(m, n; k)$ is a smooth submanifold of U. By interchanging suitable rows and columns one proves analogous assertions for neighbourhoods of other matrices in $M(m, n; k)$. ◇

(1.4.3) Example. The subset

$$S_k(\mathbb{R}^n) = \{(x_1, \dots, x_k) \mid x_i \in \mathbb{R}^n;\ x_1, \dots, x_k \text{ linearly independent}\}$$

of the k-fold product of $\mathbb{R}^n$ is called the ***Stiefel manifold*** of k-frames in $\mathbb{R}^n$. It can be identified with $M(k, n; k)$ and carries this structure of a smooth manifold. ◇

(1.4.4) Example. The group $O(n)$ of orthogonal (n,n)-matrices is a smooth submanifold of the vector space $M_n(\mathbb{R})$ of real (n,n)-matrices. Let $S_n(\mathbb{R})$ be the subspace of symmetric matrices. The map $f\colon M_n(\mathbb{R}) \to S_n(\mathbb{R}),\ B \mapsto B^t \cdot B$ is smooth, $O(n) = f^{-1}(E)$, and f has surjective differential at each point $A \in O(n)$. The derivative at $s = 0$ of $s \mapsto (A^t + sX^t)(A + sX)$ is $A^t \cdot X + X^t \cdot A$; the differential of f at A is the linear map $M_n(\mathbb{R}) \to S_n(\mathbb{R}),\ X \mapsto A^t \cdot X + X^t \cdot A$. It is surjective, since the symmetric matrix S is the image of $X = \frac{1}{2}AS$. From 1.2.3 we obtain

$$T_A O(n) = \{X \in M_n(\mathbb{R}) \mid A^t \cdot X + X^t \cdot A = 0\},$$

and in particular for the unit matrix E, $T_E O(n) = \{X \in M_n(\mathbb{R}) \mid A^t + A = 0\}$, the space of skew-symmetric matrices. A local parametrization of $O(n)$ about E can be obtained from the exponential map $T_E O(n) \to O(n), X \mapsto \exp X = \sum_0^\infty X^k/k!$. Group multiplication and passage to the inverse are smooth maps. ◇

(1.4.5) Example. The Stiefel manifolds have an orthogonal version which generalizes the orthogonal group, the ***Stiefel manifold of orthonormal k-frames***. Let $V_k(\mathbb{R}^n)$ be the set of orthonormal k-tuples $(v_1, \dots, v_k)$, $v_j \in \mathbb{R}^n$. If we write v_j as row vector, then $V_k(\mathbb{R}^n)$ is a subset of the vector space $M = M(k, n; \mathbb{R})$ of real (k,n)-matrices. Let $S = S_k(\mathbb{R})$ again be the vector space of symmetric (k,k)-matrices. Then $f\colon M \to S, A \mapsto A \cdot A^t$ has the pre-image $f^{-1}(E) = V_k(\mathbb{R}^n)$. The differential of f at A is the linear map $X \mapsto XA^t + AX^t$ and it is surjective. Hence E is a regular value. The dimension of $V_k(\mathbb{R}^n)$ is $(n-k)k + \frac{1}{2}k(k-1)$. ◇

Problems

1. Make an analysis of the unitary group $U(n)$ along the lines of 1.4.4.
2. Let $\Lambda^k(\mathbb{R}^n)$ be the k-th exterior power of $\mathbb{R}^n$. The action of $O(n)$ on $\mathbb{R}^n$ induces an action on $\Lambda^k(\mathbb{R}^n)$, a smooth representation. If we assign to a basis $x(1), \dots, x(k)$ of a k-dimensional subspace the element $x(1) \wedge \dots \wedge x(k) \in \Lambda^k(\mathbb{R}^n)$, we obtain a well-defined, injective, $O(n)$-equivariant map $j\colon G_k(\mathbb{R}^n) \to P(\Lambda^k\mathbb{R}^n)$ (***Plücker coordinates***). The image of j is a smooth submanifold of $P(\Lambda^k\mathbb{R}^n)$, i.e., j is an embedding

of the Grassmann manifold $G_k(\mathbb{R}^n)$.
3. The ***Segre embedding*** is the smooth embedding

$$\mathbb{R}P^m \times \mathbb{R}P^n \to \mathbb{R}P^{(m+1)(n+1)-1}, \quad ([x_i],[y_j]) \mapsto [x_i y_j].$$

For $m = n = 1$ the image is the quadric $\{[s_0,s_1,s_2,s_3] \mid s_0s_3 - s_1s_2 = 0\}$.
4. Let $h\colon \mathbb{R}^{n+1}\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+k+1}$ be a symmetric bilinear form such that $x \neq 0, y \neq 0$ implies $h(x,y) \neq 0$. Let $g\colon S^n \to S^{n+k}$, $x \mapsto h(x,x)/|h(x,x)|$. If $g(x) = g(y)$, hence $h(x,x) = t^2 h(y,y)$ with some $t \in \mathbb{R}$, then $h(x+ty, x-ty) = 0$ and therefore $x+ty = 0$ or $x-ty = 0$. Hence g induces a smooth embedding $\mathbb{R}P^n \to S^{n+k}$. The bilinear form $h(x_0,\dots,x_n,y_0,\dots,y_n) = (z_0,\dots,z_{2n})$ with $z_k = \sum_{i+j=k} x_i y_j$ yields an embedding $\mathbb{R}P^n \to S^{2n}$ [**?**] [**?**].
5. Let $h\colon \mathbb{R}^{n+1}\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+k+1}$ be a symmetric bilinear form such that $x \neq 0, y \neq 0$ implies $h(x,y) \neq 0$. Leti $g\colon S^n \to S^{n+k}$, $x \mapsto h(x,x)/|h(x,x)|$. If $g(x) = g(y)$, hence $h(x,x) = t^2 h(y,y)$ with some $t \in \mathbb{R}$, then $h(x+ty, x-ty) = 0$ and therefore $x+ty = 0$ or $x-ty = 0$. Hence g induces a smooth embedding $\mathbb{R}P^n \to S^{n+k}$. The bilinear form $h(x_0,\dots,x_n,y_0,\dots,y_n) = (z_0,\dots,z_{2n})$ with $z_k = \sum_{i+j=k} x_i y_j$ yields an embedding $\mathbb{R}P^n \to S^{2n}$ [**?**] [**?**].
6. Remove a point from $S^1 \times S^1$ and show (heuristically) that the result has an immersion into $\mathbb{R}^2$. (Removing a point is the same as removing a big 2-cell!).

1.5 Quotients

(1.5.1) Theorem. *Let M be a smooth n-manifold. Let $C \subset M \times M$ be the graph of an equivalence relation R on M, i.e., $C = \{(x,y) \mid x \sim y\}$. Then the following are equivalent:*
(1) *The set of equivalence classes $N = M/R$ carries the structure of a smooth manifold such that the quotient map $p\colon M \to N$ is a submersion.*
(2) *C is a closed submanifold of $M \times M$ and* $\mathrm{pr}_1\colon C \to M$ *is a submersion.*

Proof. (1) ⇒ (2). Since N is a Hausdorff space, the diagonal $D \subset N \times N$ is a closed submanifold. The product $p\times p$ is a submersion, hence $(p\times p)^{-1}(D) = C$ a closed submanifold.

Let $(x,y) \in C$. Let V be an open neighbourhood of $p(x)$ in N and $s\colon V \to M$ a local section of p with $sp(x) = y$. Then $\tau\colon p^{-1}(V) \to C$, $z \mapsto (z, sp(z))$ is a smooth map such that $\tau(x) = (x,y)$ and $\mathrm{pr}_1 \circ \tau = \mathrm{id}$. Therefore pr_1 is a submersion in a neighbourhood of (x,y).

(2) ⇒ (1). The construction of a smooth structure on N is based on the following assertions (A) and (B).

(1.5.2) Lemma. (A) *For each $x \in M$ there exists an open neighbourhood U and a retractive submersion $u\colon U \to S$ onto a submanifold S of U such that*

$$C \cap (U \times U) = \{(z_1,z_2) \in U \times U \mid u(z_1) = u(z_2)\}.$$

(B) *For each* $(x,y) \in C$ *there exists an open neighbourhood* U *of* x *in* M *and a smooth map* $s\colon U \to M$ *with* $s(x) = y$ *and:* $u \in U \Rightarrow u \sim s(u)$.

Let (U, u, S) be chosen according to (A). The left side of the equality is the restriction of the equivalence relation to U. Therefore there exists a bijection $\overline{u}\colon p(U) \to S$ such that $\overline{u} \circ p = u$. We want to define the smooth structure on N such that $\overline{u}$ is a diffeomorphism. Let (V, v, T) be a second datum according to (A). Let

$$x = p(a) = p(b) \in p(U) \cap p(V), \qquad a \in U,\, b \in V.$$

By (B) and continuity of s there exist open neighbourhoods $U_0 \subset U$ of a, $V_0 \subset V$ of b and a smooth map $s\colon U_0 \to V_0$ such that $s(a) = b$ and $s(z) \sim z$ for each $z \in U_0$. This implies $p(U_0) \subset p(V_0) \subset p(U) \cap p(V)$. The set $U_0^* = u(U_0)$ is contained in $\overline{u}(p(U) \cap p(V))$ and an open neighbourhood of $\overline{u}(x)$, since u is an open map. Hence $\overline{u}(p(U) \cap p(V))$ is open in S. We show that $\overline{v} \circ \overline{u}^{-1}$ is smooth. Since $u\colon U_0 \to U_0^*$ is a submersion, there exists (after shrinking of U_0) a smooth section $t\colon U_0^* \to U_0$ of this map, and $\overline{v} \circ \overline{u}^{-1} | U_0^* = v \circ s \circ t$ is smooth.

We have now verified the hypothesis for the gluing process: There exists a unique topology on N such that the $p(U)$ are open and the $\overline{u}\colon p(U) \to S$ homeomorphisms. By construction, p is a continuous open map. Hence $p \times p$ is open, and therefore $(p \times p)(M \times M \smallsetminus C) = N \times N \smallsetminus D$ open and N a Hausdorff space. In general, if $\mathcal{B}$ is a basis for the topology on X and $f\colon X \to Y$ a continuous, surjective, open map, then $\{f(B) \mid B \in \mathcal{B}\}$ is a basis of Y. Hence N has a countable basis.

The smooth structure on N is determined by the conditions that the maps $\overline{u}\colon p(U) \to S$ are diffeomorphisms. This also shows that p is a submersion.

Proof of (B). Let $(x,y) \in C$. Since $\mathrm{pr}_1\colon C \to M$ is a submersion, there exists an open neighbourhood U of x in M and a smooth map $\sigma\colon U \to C$ such that $\sigma(x) = (x,y)$ and $\mathrm{pr}_1 \circ \sigma = \mathrm{id}(U)$. Then $s = \mathrm{pr}_2 \circ \sigma$ satisfies (B). □

Proof of (A). The proof is subdivided into several steps.

(i) The set C contains the diagonal of $M \times M$, hence C has dimension $m + n$, $0 \le m \le n$. Let $x \in M$. There exists an open neighbourhood U_0 of x in M and a map $f\colon U_0 \times U_0 \to \mathbb{R}^{n-m}$ of constant rank $n - m$ such that $C \cap (U_0 \times U_0) = \{(z, z') \in U_0 \times U_0 \mid f(z, z') = 0\}$.

(ii) We claim that $f_l\colon U_0 \to \mathbb{R}^{n-m}$, $z \mapsto f(x, z)$ has a surjective differential at x. Note that the diagonal Δ_x of $T_x U_0 \times T_x U_0$ is contained in $T_{(x,x)} C$, since C contains the diagonal of M. Since $T_{(x,x)} f$ is surjective and $T_{(x,x)} C$, hence Δ_x, is contained in the kernel of $T_{(x,x)} f$, we obtain an induced surjective map

$$T_x f_l\colon T_x U_0 \cong (T_x U_0 \times T_x U_0)/\Delta_x \to \mathbb{R}^{n-m}, \quad v \mapsto (0, v) \mapsto T_{(x,x)} f(0, v).$$

In a similar manner we see that $f_r\colon z \mapsto f(z, x)$ has a surjective differential at x.

(iii) We choose by the rank theorem a smooth map $g\colon U_0 \to \mathbb{R}^m$ with $g(x) = 0$ such that

$$F = (f_l, g)\colon U_0 \to \mathbb{R}^{n-m} \times \mathbb{R}^m, \quad z \mapsto (f(x,z), g(z))$$

maps U_0 (perhaps after shrinking) diffeomorphically onto an open set.

(iv) Let $h\colon U_0 \times U_0 \to \mathbb{R}^{n-m} \times \mathbb{R}^m$, $(z, z') \mapsto (f(z,z'), g(z'))$. The partial map $F = h(x, ?)\colon z' \mapsto h(x, z')$ has bijective differential at x, by (iii). Hence there exist open neighbourhoods W and W' of x in U_0 and a smooth map $u\colon W \to W'$ such that

$$\{(z, z') \in W \times W' \mid f(z, z') = 0, g(z') = 0\} = \{(z, u(z)) \mid z \in W\}.$$

By the choice of g and U_0 in (iii), $B = g^{-1}(0) \subset U_0$ is a submanifold, and u is, by construction, a map $u\colon W \to W' \cap B$ which yields a point $u(z) \in B$ in the equivalence class of z.

(v) We show: The differential $T_x u\colon T_x W \to T_x(W' \cap B)$ is surjective. Since $h(z, u(z)) = 0$ the relation

$$T_x u = -T_x h(x, ?)^{-1} \circ T_x h(?, x)$$

holds. Since $h(?, x)$ equals f_r (up to a zero component), the rank of $T_x h(?, x)$ equals the rank of $T_x f_r$; hence this rank is $n - m$, and this is the dimension of $T_x(W' \cap B)$ and the rank of $T_x u$.

(vi) If we shrink W, we do not affect (iv). We therefore choose W small enough such that $u\colon W \to W' \cap B$ has constant rank. Let $T = u(W)$ and $z \in T \cap W \subset W' \cap B = u(W) \cap W \subset u(W) \subset W'$. Then $(z, z) \in W \times W'$ and hence $u(z) = z$. Let $U = u^{-1}(T \cap W)$ and $S = U \cap T$. We show: $u(U) \subset S$. So let $z \in U$. Then we know that

$$u(z) \in u(U) = u(u^{-1}(T \cap W)) \subset T \cap W$$

and, by what we have already proved, $u(u(z)) = u(z) \in T \cap W$, i.e., $u(z) \in u^{-1}(T \cap W) = U$; moreover $u(z) \in T \cap W$ and hence altogether $u(z) \in U \cap T = S$.

We have now obtained an open neighbourhood U of x in M and a submersive retraction $u\colon U \to S$ onto a submanifold S of U such that

$$C \cap (U \times S) = \{(z, u(z)) \mid z \in U\}.$$

(vii) Let $(z_1, z_2) \in C \cap (U \times U)$. Then $(z_1, u(z_1)) \in C$ and $(z_2, u(z_2)) \in C$. Since C is an equivalence relation, we conclude that $(u(z_1), u(z_2)) \in C \cap (S \times S)$ and hence $u(z_1) = u(z_2)$. Finally we see

$$C \cap (U \times U) = \{(z_1, z_2) \in U \times U \mid u(z_1) = u(z_2)\}.$$

This finishes the proof of (A). □

1.6 Smooth Transformation Groups

Let G be a Lie group and M a smooth manifold. We consider smooth action $G \times M \to M$ of G on M. The left translations $l_g\colon M \to M, x \mapsto gx$ are then diffeomorphisms. The map $\beta\colon G \to M, g \mapsto gx$ is a smooth G-map with image the orbit $B = Gx$ through x. We have an induced bijective G-equivariant set map $\gamma\colon G/G_x \to B$. The map β has constant rank; this follows from the equivariance. If $L_g\colon G \to G$ and $l_g\colon M \to M$ denote the left translations by g, then $l_g\beta = \beta L_g$, and since L_g and l_g are diffeomorphisms, we see that $T_e\beta$ and $T_g\beta$ have the same rank.

(1.6.1) Proposition. *Suppose the orbit $B = Gx$ is a smooth submanifold of M. Then:*

(1) *$\beta\colon G \to B$ is submersion.*
(2) *$G_x = \beta^{-1}(x)$ is a closed Lie subgroup of G.*
(3) *There exists a unique smooth structure on G/G_x such that the quotient map $G \to G/G_x$ is a submersion. The induced map $\gamma\colon G/G_x \to B$ is a diffeomorphism.*

Proof. If β would have somewhere a rank less than the dimension of B, the rank would always be less than the dimension, by equivariance. This contradict the theorem of Sard. We transport via γ the smooth structure from B to G/G_x. The smooth structure is unique, since $G \to G/G_x$ is a submersion. The pre-image G_x of a regular value is a closed submanifold. □

The previous proposition gives us G_x as a closed Lie subgroup. We need not use the general theorem about closed subgroups being Lie subgroups.

(1.6.2) Example. The action of $SO(n)$ on S^{n-1} by matrix multiplication is a smooth action. We obtain a resulting equivariant diffeomorphism $S^{n-1} \cong SO(n)/SO(n-1)$. In a similar we obtain equivariant diffeomorphisms we have $S^{2n-1} \cong U(n)/U(n-1) \cong SU(n)/SU(n-1)$. ◇

(1.6.3) Theorem. *Let M be a smooth G-manifold with free, proper action of the Lie group G. Then the orbit space M/G carries a smooth structure and the orbit map $p\colon M \to M/G$ is a submersion.*

Proof. We verify the hypothesis of the quotient theorem **??**. We have to show that C is a closed submanifold. The set C is homeomorphic to the image of the map $\Theta\colon G \times M \to M \times M, (g,x) \mapsto (x, gx)$, since the action is proper. We show that Θ is a smooth embedding. It suffices to show that Θ is an immersion (1.2.5). The differential

$$T_{(g,x)}\Theta\colon T_gG \times T_xM \to T_xM \times T_{gx}M$$

will be decomposed according to the two factors

$$T_{(g,x)}\Theta(u,v) = T_g\Theta(?,x)u + T_x\Theta(g,?)v.$$

The first component of $T_g\Theta(?,x)u$ is zero, since the first component of the partial map is constant. Thus if $T_{(g,x)}(u,v) = 0$, the component of $T_x\Theta(g,?)v$ in T_xM is zero; but this component is v. It remains to show that $T_gf\colon T_gG \to T_{gx}M$ is injective for $f\colon G \to M, g \mapsto gx$. Since the action is free, the map f is injective; and f has constant rank, by equivariance. An injective map of constant rank has injective differential, by the rank theorem. Thus we have verified the first hypothesis of **??**. The second one holds, since $\mathrm{pr}_1 \circ \Theta = \mathrm{pr}_2$ shows that pr_1 is a submersion. □

(1.6.4) Example. The cyclic group $G = \mathbb{Z}/m \subset S^1$ acts on $\mathbb{C}^n$ by

$$\mathbb{Z}/m \times \mathbb{C}^n \to \mathbb{C}^n, \quad (\zeta, (z_1, \ldots, z_n)) \mapsto (\zeta^{r_1} z_1, \ldots, \zeta^{r_n} z_n)$$

where $r_j \in \mathbb{Z}$. This action is a smooth representation. Suppose the integers r_j are co-prime to m. The induced action on the unit sphere is then a free G-manifold $S(r_1, \ldots, r_n)$; the orbit manifold $L(r_1, \ldots, r_n)$ is called a (generalized) ***lens space***. ◇

(1.6.5) Example. Let H be a closed Lie subgroup of the Lie group G. The H-action on G by left translation is smooth and proper. The orbit space $H\backslash G$ carries a smooth structure such that the quotient map $G \to H\backslash G$ is a submersion. The G-action on $H\backslash G$ is smooth. One can consider the projective spaces, Stiefel manifolds and Grassmann manifolds as homogeneous spaces from this view point. ◇

(1.6.6) Theorem. *Let M be a smooth G-manifold. Then:*

(1) *An orbit $C \subset M$ is a smooth submanifold if and only if it is a locally closed subset.*
(2) *If the orbit C is locally closed and $x \in C$, then there exists a unique smooth structure on G/G_x such that the orbit map $G \to G/G_x$ is a submersion. The map $G/G_x \to C$, $gG_x \mapsto gx$ is a diffeomorphism. The G-action on G/G_x is smooth.*
(3) *If the action is proper, then* (1) *and* (2) *hold for each orbit.*

Proof. (1) $\beta\colon G \to C, g \mapsto gx$ has constant rank by equivariance. Hence there exists an open neighbourhood of e in G such that $\beta(U)$ is a submanifold of M. Since C is locally closed in the locally compact space M, the set C is locally compact and therefore $\beta\colon G \to C$ an open map (see **??**). Hence there exists an open set W in M such that $C \cap W = \beta(U)$. Therefore C is a submanifold in a neighbourhood of x and, by equivariance, also globally a submanifold.

(2) Since C is locally closed, the submanifold C has a smooth structure. The map β has constant rank and is therefore a submersion. We now transport the smooth structure from C to G/G_x.

(3) The orbits of a proper action are closed. □

1.7 Slice Representations

(1.7.1) Proposition. *Let M be a smooth G-manifold. Let $x \in M$ be a point with compact isotropy group G_x. Then there exist a G_x-equivariant chart (W, ψ, T_xM) centered in x.*

Proof. We start with an arbitrary chart (U, φ, T_xM) which is centered at x. The orbit map $p\colon M \to M/H$ is closed, since $H = G_x$ is compact. Hence $W = M \smallsetminus p^{-1}p(M \smallsetminus U)$ is open, H-invariant and contained in U. We use the invariant integration on H: A linear map $\int\colon C(H, \mathbb{R}) \to \mathbb{R}, f \mapsto \int f(h)\,dh$ from the vector space of continuous functions $H \to \mathbb{R}$ which maps the constant function 1 to 1 and has the property $\int f(hu)\,dh = \int f(uh)\,dh = \int f(h)\,dh$ for each $u \in H$. We define

$$\psi\colon W \to T_xM, \quad z \mapsto \int_H h^{-1} \cdot \varphi(hz)\,dh$$

with the H-action $(h, v) \mapsto h \cdot v$ on T_xM given by the differential of the action on M. By invariance of the integral, ψ is H-equivariant. After a suitable restriction to a smaller W we can use ψ as a chart. For the proof we show that the differential of ψ at x is bijective. For this purpose we start with a chart φ such that $T_x\varphi = \mathrm{id}(T_xM)$. We claim: The differential of ψ is the identity. This is seen by differentiating "under the integral symbol", since $z \mapsto h \cdot \varphi(hz)$ has at x the identity as differential. □

(1.7.2) Corollary. *Let G be compact. Then the fixed point set M^G is a smooth submanifold.*

Proof. In a neighbourhood of $x \in M^G$ the manifold M is G-diffeomorphic to a representation V (2.8.3). In a representation the fixed point set is a linear subspace, thus we obtain an adapted chart. □

Let $G_x = H$ be compact. Suppose the orbit C through x is a submanifold. Then T_xC is an H-stable subspace of T_xM. Let V be an H-stable complement of T_xC in T_xM; it is uniquely determined as an H-representation, up to isomorphism. We call V the ***slice representation*** of M in x.

Let $\varphi\colon U \to T_xM$ be an H-equivariant chart with inverse ψ. The map

$$\tilde{\tau}\colon G \times V \to M, \quad (g, v) \mapsto g\psi(v)$$

factors over the orbit space $G \times_H V$. Since H is compact, $G \times_H V$ is in a canonical manner a smooth G-manifold, and $\tilde{\tau}$ induces a smooth G-map $\tau\colon G \times_H V \to M$.

We choose an H-invariant inner product on V and set

$$V(\varepsilon) = \{v \in V \mid \|v\| < \varepsilon\}.$$

(1.7.3) Proposition. *There exists $\varepsilon > 0$ such that $\tau\colon G \times_H V(\varepsilon) \to M$ is an embedding onto an open neighbourhood of C (equivariant tubular map).*

Proof. We begin by showing that τ has bijective differential at points of the zero section. This is a consequence of the relations

$$T_{(e,0)}\tilde{\tau}(T_e G \times \{0\}) = T_x C, \quad T_{(e,0)}\tilde{\tau}(\{e\} \times V) = V$$

and the fact that τ is a G-map between manifolds of the same dimension. For each compact set $L \subset G/H$ there exists an $\epsilon > 0$ such that τ embeds $p^{-1}(L) \times_H V(\epsilon)$. Since the action is proper (2.8.3), there exists an $\eta > 0$ such that

$$\{g \in G \mid g\psi(V(\eta)) \cap \psi(V(\eta) \neq \emptyset\}$$

is contained in a compact set K. We choose an open neighbourhood W of eH in G/H with compact closure andϵ with $0 < \epsilon < \eta$ such that $p^{-1}(W) \supset KH$ and τ embeds $p^{-1}(W) \times_H V(\epsilon)$ We claim that then also $G \times_H V(\epsilon)$ is embedded. By equivariance, τ has bijective differential everywhere and is therefore an open map. Thus it suffices to show that τ is injective. But $g_1\psi(v_1) = g_2\psi(v_2)$ implies $g_2^{-1}g_1\psi(v_1) = \psi(v_2)$, hence $k = g_2^{-1}g_1 \in K$. The points (k, v_1) and (e, v_2) are contained in $p^{-1}(W) \times V(\epsilon)$. We conclude $k \in H$ and $kv_1 = v_2$; hence (g_1, v_1) and (g_2, v_2) represent the same element in $G \times_H V(\epsilon)$. □

1.8 Principal Orbits

In this section we study smooth G-manifolds for a compact group G.

(1.8.1) Theorem. *A compact G-manifold M has finite orbit type, i.e., the set of conjugacy classes of isotropy groups is finite.*

Proof. Induction on $\dim M$. For $\dim M = 0$ the situation is clear. By compactness, M has a finite covering by sets of the form $G \times_H V$ with orthogonal H-representations V. The unit sphere SV has smaller dimension and finite H-orbit type by induction hypothesis.

The isotropy group of $(u, v) \in G \times_H V$ consists of the $g \in G$ such that the relation $(gu, v) = uh, h^{-1}v$ holds for some $h \in H$. This means $h \in H_v$ and

$g \in uH_v u^{-1}$. Therefore an isotropy group of $G \times_H V$ is G-conjugate to an H-isotropy group of V. The H-isotropy groups of$V$ are H and the isotropy groups of SV. □

(1.8.2) Theorem (Principal Orbit). *Let M/G be connected. Then the following holds:*

(1) *There exists a unique isotropy type (H) of M such that the orbit bundle $M_{(H)}$ is open and dense in M.*
(2) *The space $M_{(H)}/G$ is connected.*
(3) *Each isotropy type (K) of M is subconjugate to (H).*
(4) *M^H intersects each orbit.*

The orbit type (H) in the previous theorem is called the ***principal orbit type*** of M and $M_{(H)}$ the associated ***principal orbit bundle***.

Proof. Induction on $\dim M$. In the case that $\dim M = 0$ the connectedness of M/G means that M/G is a point and hence M a single orbit.

Suppose that $\dim M \geq 1$.

(1) We begin with manifolds of the form $G \times_H V$. Since $(G \times_H SV)/G \cong SV/H$ we see that for a non-connected orbit space $G \times_H SV$ necessarily $\dim V = 1$ and V is the trivial H-representation. In that case $G \times_H V \cong G/H \times \mathbb{R}$ and the theorem holds for this space.

Thus let $G \times_H SV$ be connected. By induction, the assertions of the theorem hold for this manifold and also for the H-manifold SV. Let $K \subset H$ and (K) by the principal isotropy type of SV. Then either $0 \in V_{(K)}$ and hence $H = K$, or $V_{(K)} \cong SV_{(K)} \times]0, \infty[$.

In the first case $SV_{(H)} = SV^H$; and since $SV_{(H)}$ is dense in SV, we conclude $SV = SV^H$ and this means that the representation V is trivial. But for $G \times_H V \cong G/H \times V$ the theorem holds.

In the second case

$$(G \times_H V)_{(K)} = G \times_H V_{(K)},$$

and since $V_{(K)}$ is open and dense in $V \setminus 0$ and V, the set $(G \times_H V)_{(K)}$ is open and dense in $G \times_H V$. The set

$$(G \times_H V)_{(K)}/G \cong V_{(K)}/H \cong SV_{(K)} \times]0, \infty[$$

is connected. Since two open and dense subsets have non-empty intersection, there exists at most one isotropy type (H) for which the first statement of the theorem holds. Hence the first two assertions hold for K in place of H for $M = G \times_H V$. This settles the case under consideration.

(2) We now cover M with G-sets of the form $G \times_H V$ for suitable H and V. If two such G-sets have non-empty intersection, then also their principal orbit bundles as open an dense subsets, hence the corresponding isotropy types

coincide. Since M/G is connected, the isotropy types of each of the subsets coincide, and the union of their orbit bundles is then open and dense in M.

The union of connected sets with non-empty intersection is connected. Hence $M_{(H)}/G$ is connected.

(3) Let now K be any isotropy group, $K = G_x$. Let U a neighbourhood of the orbit x which is isomorphic to $G \times_K V$. By denseness, there exists an orbit in U which is contained in the principal orbit bundle This orbit has via the projection $G \times_K V \to G/K$ an equivariant map into G/K, hence $(H) \le (K)$. We conclude $G/K^H \neq \emptyset$, hence $(Gx)^H \neq \emptyset$, and finally $M^H \cap Gx \neq \emptyset$. □

(1.8.3) Proposition. *Let H be an isotropy group of the G-manifold M.*

1. *The orbit bundle $M_{(H)}$ is a submanifold (perhaps with components of different dimension).*
2. *The closure $\overline{M}_{(H)}$ only contains smaller orbits, i.e., orbits which admit a G-map $G/H \to G/K$.*
3. *The set $M_{(H)}$ is open in its closure.*

Proof. (1) A local consideration suffices. So let us assume $M = G \times_H V$. From $G_{(g,v)} = gH_vg^{-1}$ we see that $(G_{(g,v)}) = (H)$ if and only if $H_v = H$ and $v \in V^H$. The set

$$(G \times_H V)_{(H)} = G \times_H V^H \subset G \times_H V$$

is a smooth subbundle, hence a smooth submanifold.

(2) If $z \in M$ and $K = G_z$, then there exists a neighbourhood W of z of the form $G \times_K W$, and in a neighbourhood of this type $(G_y) \le (K)$ holds for each $y \in W$.

If $z \in \overline{M}_{(H)}$, then W contains points y with $(G_y) = (H)$. Hence $(H) \le (G_z)$.

(3) Let $x \in M_{(H)}$ and U a neighbourhood with $(G_y) \le (H)$ for $y \in U$. If $z \in U \cap \overline{M}_{(H)}$, then $(H) \le (G_z) \le (H)$; therefore $U \cap \overline{M}_{(H)} \subset M_{(H)}$, i.e., $M_{(H)}$ is open in its closure. □

(1.8.4) Proposition. *Let M/G be connected. If an orbit of type G/U has smaller dimension that a principal orbit, then $M_{(U)}$ has at least codimension 2 in M.*

Proof. Induction on $\dim M$. In the case $\dim M = 0$ the set $M_{(U)}$ is empty. For the induction step it suffices to consider $M = G \times_H V$. We have already shown that

$$(G \times_U V)_{(U)} \cong G \times_U V^U \cong G/U \times V^U.$$

Thus we have to show that V^U has at least codimension 2 in V. Consider the U-manifold SV. Suppose $SV = \emptyset$, and hence $\dim V = 0$, then $M = G/U$ and (U) would be the principal isotropy type. Let $\dim SV = 0$, hence $\dim G/U = n - 1 = \dim M - 1$. Then a principal orbit must have dimension n, and then M would consist of a single orbit.

Hence we can assume $\dim V \geq 2$. Then SV and SV/U are connected. We apply the induction hypothesis to the U-manifold SV. The fixed point set SV^U has a smaller dimension than the principal orbit, since otherwise the orbits in SV an hence V would be 0-dimensional. Then the Die G-orbits of $G \times_U V$ would have the dimension of G/U, in contradiction to our assumption about U. Hence we can apply to SV^U the induction hypothesis: $\dim SV^U \leq \dim SV - 2$. □

(1.8.5) Proposition. *Let the compact Lie group G acts smoothly and effectively on the connected n-manifold M. Then*

$$\dim G \leq \frac{1}{2} n(n+1).$$

Proof. Induction on n. In the case $n = 0$ the manifold M is a point, hence $M = G/G$, and therefore $G = \{e\}$, since the action is effective.

We can assume that G is connected, since the component of the neutral element acts effectively and has the same dimension.

Let G/H be a principal orbit, hence $\dim G \leq n + \dim H$. If H acts effectively on the connected k-manifold with $k \leq n-1$, then we can apply the induction hypothesis.

We have $k = \dim G/H \leq n$, and G acts effectively on G/H. Namely if $g \in G$ would acts trivially on G/H, the also trivially on the principal orbit bundle, hence by denseness trivially at all. Therefore the component H_0 of e in H acts effectively on G/H and also on the principal orbit bundle of the H_0-manifold G/H. The principal orbit is connected. If the dimensions of the principal orbit is less that k we are ready.

In the the remaining case $H_0 eH = G/H$, and G/H is a point, hence $G = \{e\}$. □

(1.8.6) Example. The group $SO(n+1)$ has dimension $(n+1)n/2$ and acts effectively on S^n and $\mathbb{R}P^n$. ◇

1.9 Manifolds with Boundary

We now extend the notion of a manifold to that of a manifold with boundary. A typical example is the n-dimensional disk $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. Other examples are half-spaces. Let $\lambda\colon \mathbb{R}^n \to \mathbb{R}$ be a non-zero linear form. We use the corresponding ***half-space*** $H(\lambda) = \{x \in \mathbb{R}^n \mid \lambda(x) \geq 0\}$. Its ***boundary*** $\partial H(\lambda)$ is the kernel of λ. Typical half-spaces are $\mathbb{R}^n_\pm = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \pm x_1 \geq 0\}$. If $A \subset \mathbb{R}^m$ is any subset, we call $f\colon A \to \mathbb{R}^n$ ***differentiable*** if for each $a \in A$ there exists an open neighbourhood U of a in $\mathbb{R}^m$ and a differentiable map $F\colon U \to \mathbb{R}^n$ such that $F|U \cap A = f|U \cap A$. We only apply this definition to open subset A of half-spaces. In that case, the differential of F at $a \in A$ is independent of the choice of the extension F and will be denoted $Df(a)$.

Let $n \geq 1$ be an integer. An ***n-dimensional manifold with boundary*** or ***∂-manifold*** is a Hausdorff space M with countable basis such that each point has an open neighbourhood which is homeomorphic to an open subset in a half-space of $\mathbb{R}^n$. A homeomorphism $h\colon U \to V$, U open in M, V open in $H(\lambda)$ is called a ***chart*** about $x \in U$ with ***chart domain*** U. With this notion of chart we can define the notions: C^k-related, atlas, differentiable structure. An ***n-dimensional smooth manifold with boundary*** is therefore an n-dimensional manifold M with boundary together with a (maximal) smooth C^∞-atlas on M.

Let M be a manifold with boundary. Its ***boundary*** ∂M is the following subset: The point x is contained in ∂M if and only if there exists a chart (U, h, V) about x such that $V \subset H(\lambda)$ and $h(x) \in \partial H(\lambda)$. The complement $M \smallsetminus \partial M$ is called the ***interior*** $\mathrm{In}(M)$ of M. The following lemma shows that specifying a boundary point does not depend on the choice of the chart (invariance of the boundary).

(1.9.1) Lemma. *Let $\varphi\colon V \to W$ be a diffeomorphism between open subsets $V \subset H(\lambda)$ and $W \subset H(\mu)$ of half-spaces in $\mathbb{R}^n$. Then $\varphi(V \cap \partial H(\lambda)) = W \cap \partial H(\mu)$.* □

(1.9.2) Proposition. *Let M be an n-dimensional smooth manifold with boundary. Then either $\partial M = \emptyset$ or ∂M is an $(n-1)$-dimensional smooth manifold. The set $M \smallsetminus \partial M$ is a smooth n-dimensional manifold with empty boundary.*

Proof. Let $\partial M \neq \emptyset$. The assertion about ∂M means that the differential structure on M induces a differential structure on ∂M in the following manner. A little thinking shows that M has an atlas which consists of charts (U, h, V) with V open in $\mathbb{R}^n_-$, called ***adapted to the boundary***. The charts for ∂M are the restrictions $h\colon U \cap \partial M \to V \cap \partial \mathbb{R}^n_-$ of such charts (they form an atlas). Then $V \cap \partial \mathbb{R}^n_-$ is open in $0 \times \mathbb{R}^{n-1} \cong \mathbb{R}^{n-1}$. The charts for $M \smallsetminus \partial M$ are the restrictions $h\colon U \cap (M \smallsetminus \partial M) \to V \cap (\mathbb{R}^n_- \smallsetminus \partial \mathbb{R}^n_-)$. The latter set is open in $\mathbb{R}^n$. □

The boundary of a manifold can be empty. Sometimes it is convenient to view the empty set as an n-dimensional manifold. If $\partial M = \emptyset$, we call M a manifold without boundary. This coincides then with the notion introduced in the first section. In order to stress the absence of a boundary, we call a compact manifold without boundary a ***closed manifold***.

A map $f\colon M \to N$ between smooth manifolds with boundary is called ***smooth*** if it is continuous and C^∞-differentiable in local coordinates. ***Tangent spaces*** and the ***differential*** are defined as for manifolds without boundary.

Let $x \in \partial M$ and $k = (U, h, V)$ be a chart about x with V open in $\mathbb{R}^n_-$. Then the pair (k, v), $v \in \mathbb{R}^n$ represents a vector w in the tangent space $T_x M$. We say that w is ***pointing outwards*** (***pointing inwards, tangential to*** ∂M) if $v_1 > 0$ ($v_1 < 0$, $v_1 = 0$, respectively). One verifies that this disjunction is independent of the choice of charts.

(1.9.3) Proposition. *The inclusion* $j\colon \partial M \subset M$ *is smooth and the differential* $T_x j\colon T_x(\partial M) \to T_x M$ *is injective. Its image consists of the vectors tangential to* ∂M*. We consider* $T_x j$ *as an inclusion.* □

The notion of a submanifold can have different meanings for manifolds with boundary. We define therefore submanifolds of type I and type II.

Let M be a smooth n-manifold with boundary. A subset $N \subset M$ is called a k-dimensional smooth ***submanifold*** (of type I) if the following holds: For each $x \in N$ there exists a chart (U, h, V), $V \subset \mathbb{R}^n_-$ open, of M about x such that $h(U \cap N) = V \cap (\mathbb{R}^k \times 0)$. Such charts of M are ***adapted*** to N. The set $V \cap (\mathbb{R}^k \times 0) \subset \mathbb{R}^k_- \times 0 = \mathbb{R}^k_-$ is open in $\mathbb{R}^k_-$. A diffeomorphism onto a submanifold of type I is an embedding of type I. From this definition we draw the following conclusions.

(1.9.4) Proposition. *Let* $N \subset M$ *be a smooth submanifold of type* I*. The restrictions* $h\colon U \cap N \to h(U \cap N)$ *of the charts* (U, h, V) *adapted to* N *form a smooth atlas for* N *which makes* N *into a smooth manifold with boundary. The relation* $N \cap \partial M = \partial N$ *holds, and* ∂N *is a submanifold of* ∂M*.* □

Let M be a smooth n-manifold without boundary. A subset $N \subset M$ is a k-dimensional smooth ***submanifold*** (of type II) if the following holds: For each $x \in N$ there exists a chart (U, h, V) of M about x such that $h(U \cap N) = V \cap (\mathbb{R}^k_- \times 0)$. Such charts are ***adapted*** to N.

The intersection of D^n with $\mathbb{R}^k \times 0$ is a submanifold of type I ($k < n$). The subset D^n is a submanifold of type II of $\mathbb{R}^n$. The next two propositions provide a general means for the construction of such submanifolds.

(1.9.5) Proposition. *Let* M *be a smooth* n*-manifold with boundary. Let* $f\colon M \to \mathbb{R}$ *be smooth with regular value* 0*. Then* $f^{-1}[0, \infty[$ *is a smooth submanifold of type* II *of* M *with boundary* $f^{-1}(0)$*.*

Proof. We have to show that for each $x \in f^{-1}[0, \infty[$ there exists a chart which is adapted to this set. If $f(x) > 0$, then x is contained in the open submanifold $f^{-1}]0, \infty[$; hence the required charts exist. Let therefore $f(x) = 0$. By the rank theorem 1.2.2, f has in suitable local coordinates the form $(x_1, \dots, x_n) \mapsto x_1$. From this fact one easily obtains the adapted charts. □

(1.9.6) Proposition. *Let* $f\colon M \to N$ *be smooth and* $y \in f(M) \cap (N \smallsetminus \partial N)$ *a regular value of* f *and* $f|\partial M$*. Then* $P = f^{-1}(y)$ *is a smooth submanifold of type* I *of* M *with* $\partial P = (f|\partial M)^{-1}(y) = \partial M \cap P$*.*

Proof. Being a submanifold of type I is a local property and invariant under diffeomorphisms. Therefore it suffices to consider a local situation. Let therefore U be open in $\mathbb{R}^m_-$ and $f\colon U \to \mathbb{R}^n$ a smooth map which has $0 \in \mathbb{R}^n$ as regular value for f and $f|\partial U$ $(n \geq 1, m > n)$.

We know already that $f^{-1}(0) \cap \mathrm{In}(U)$ is a smooth submanifold of $\mathrm{In}(U)$. It remains to show that there exist adapted charts about points $x \in \partial U$. Since x is a regular point of $f|\partial U$, the Jacobi-matrix $(D_i f_j(x) \mid 2 \leq i \leq m,\ 1 \leq j \leq n)$ has rank n. By interchange of the coordinates $x_2, \ldots, x_m$ we can assume that the matrix

$$(D_i f_j(x) \mid m - n + 1 \leq i \leq n, \quad 1 \leq j \leq n)$$

has rank n. (This interchange is a diffeomorphism and therefore harmless.) Under this assumption, $\varphi\colon U \to \mathbb{R}^m_-$, $u \mapsto (u_1, \ldots, u_{m-n}, f_1(u), \ldots, f_n(u))$ has bijective differential at x and therefore yields, by part (1) of the rank theorem applied to an extension of f to an open set in $\mathbb{R}^m$, an adapted chart about x. □

If only one of the two manifolds M and N has a non-empty boundary, say M, then we define $M \times N$ as the manifold with boundary which has as charts the products of charts for M and N. In that case $\partial(M \times N) = \partial M \times N$. If both M and N have a boundary, then there appear "corners" along $\partial M \times \partial N$; later we shall explain how to define a differentiable structure on the product in this case.

Problems

1. The map

$$D_n(+) = \{(x,t) \mid t > 0, \|x\|^2 + t^2 \leq 1\} \to]-1, 0] \times U_1(0), (x,t) \mapsto \left(\frac{t}{\sqrt{1-\|x\|^2}} - 1, x\right)$$

is an adapted chart for $S^{n-1} = \partial D^n \subset D^n$.

2. Let B be a ∂-manifold. A smooth function $f\colon \partial B \to \mathbb{R}$ has a smooth extension to B. A smooth function $g\colon A \to \mathbb{R}$ from a submanifold A of type I or of type II of B has a smooth extension to B.

3. Verify the invariance of the boundary for topological manifolds (use local homology groups).

4. A ∂-manifold M is connected if and only if $M \smallsetminus \partial M$ is connected.

5. Let M be a ∂-manifold. There exists a smooth function $f\colon M \to [0, \infty[$ such that $f(\partial M) = \{0\}$ and $T_x f \neq 0$ for each $x \in \partial M$.

6. Let $f\colon M \to \mathbb{R}^k$ be an injective immersion of a compact ∂-manifold. Then the image is a submanifold of type II.

7. Verify that "pointing inwards" is well-defined, i.e., independent of the choice of charts.

8. Unfortunately is not quite trivial to classify smooth 1-dimensional manifolds by just starting from the definitions. The reader may try to show that a connected

1-manifold without boundary is diffeomorphic to $\mathbb{R}^1$ or S^1; and a ∂-manifold is diffeomorphic to $[0,1]$ or $[0,1[$.

1.10 Orientation

Let V be an n-dimensional real vector space. Ordered bases $b_1,\dots,b_n$ and $c_1,\dots,c_n$ of V are called positively related if the determinant of the transition matrix is positive. This relation is an equivalence relation on the set of bases with two equivalence classes. An equivalence class is an ***orientation*** of V. We specify orientations by their representatives. The ***standard orientation*** of $\mathbb{R}^n$ is given by the standard basis $e_1,\dots,e_n$, the rows of the unit matrix. Let W be a complex vector space with complex basis $w_1,\dots,w_n$. Then $w_1, iw_1,\dots,w_n, iw_n$ defines an orientation of the underlying real vector space which is independent of the choice of the basis. This is the orientation ***induced by the complex structure***. Let $u_1,\dots,u_m$ be a basis of U and $w_1,\dots,w_n$ a basis of W. In a direct sum $U \oplus W$ we define the ***sum orientation*** by $u_1,\dots,u_m,w_1,\dots,w_n$. If $o(V)$ is an orientation of V, we denote the ***opposite orientation*** (the occidentation) by $-o(V)$. A linear isomorphism $f\colon U \to V$ between oriented vector spaces is called ***orientation preserving*** or ***positive*** if for the orientation $u_1,\dots,u_n$ of U the images $f(u_1),\dots,f(u_n)$ yield the given orientation of V.

Let M be a smooth n-manifold with or without boundary. We call two charts ***positively related*** if the Jacobi-matrix of the coordinate change has always positive determinant. An atlas is called ***orienting*** if any two of its charts are positively related. We call M ***orientable***, if M has an orienting atlas. An orientation of a manifold is represented by an orienting atlas; and two such define the same orientation if their union contains only positively related charts. If M is oriented by an orienting atlas, we call a chart ***positive*** with respect to the given orientation if it is positively related to all charts of the orienting atlas. These definitions apply to manifolds of positive dimension. An orientation of a zero-dimensional manifold M is a function $\epsilon\colon M \to \{\pm 1\}$.

Let M be an oriented n-manifold. There is an induced orientation on each of its tangent spaces T_xM. It is specified by the requirement that a positive chart (U,h,V) induces a positive isomorphism $T_xh\colon T_xM \to T_{h(x)}V = \mathbb{R}^n$ with respect to the standard orientation of $\mathbb{R}^n$. We can specify an orientation of M by the corresponding orientations of the tangent spaces.

If M and N are oriented manifolds, the ***product orientation*** on $M \times N$ is specified by declaring the products $(U \times V, k \times l, U' \times V')$ of positive charts (U,k,U') of M and (V,l,V') of N as positive. The canonical isomorphism $T_{(x,y)}(M \times N) \cong T_xM \oplus T_yN$ is then compatible with the sum orientation of vector spaces. If N is a point, then the canonical identification $M \times N \cong M$ is orientation preserving if and only if $\epsilon(N) = 1$. If M is oriented, then we denote the manifold with the opposite orientation by $-M$.

Let M be an oriented manifold with boundary. For $x \in \partial M$ we have a direct decomposition $T_x(M) = N_x \oplus T_x(\partial M)$. Let $n_x \in N_x$ be pointing outwards. The ***boundary orientation*** of $T_x(\partial M)$ is defined by that orientation $v_1, \dots, v_{n-1}$ for which $n_x, v_1, \dots, v_{n-1}$ is the given orientation of $T_x(M)$. These orientations correspond to the ***boundary orientation*** of ∂M; one verifies that the restriction of positive charts for M yield an orienting atlas for ∂M.

In $\mathbb{R}^n_-$, the boundary $\partial \mathbb{R}^n_- = 0 \times \mathbb{R}^{n-1}$ inherits the orientation defined by $e_2, \dots, e_n$. Thus positive charts have to use $\mathbb{R}^n_-$.

Let $D^2 \subset \mathbb{R}^2$ carry the standard orientation of $\mathbb{R}^2$. Consider S^1 as boundary of D^2 and give it the boundary orientation. An orienting vector in $T_x S^1$ is then the velocity vector of a counter-clockwise rotation. This orientation of S^1 is commonly known as the positive orientation. In general if $M \subset \mathbb{R}^2$ is a two-dimensional submanifold with boundary with orientation induced from the standard orientation of $\mathbb{R}^2$, then the boundary orientation of the curve ∂M is the velocity vector of a movement such that M lies "to the left".

Let M be an oriented manifold with boundary and N an oriented manifold without boundary. Then product and boundary orientation are related as follows

$$o(\partial(M \times N)) = o(\partial M \times N), \quad o(\partial(N \times M)) = (-1)^{\dim N} o(N \times \partial M).$$

The unit interval $I = [0, 1]$ is furnished with the standard orientation of $\mathbb{R}$. Since the outward pointing vector in 0 yields the negative orientation, we specify the orientation of ∂I by $\epsilon(0) = -1$, $\epsilon(1) = 1$. We have $\partial(I \times M) = 0 \times M \cup 1 \times M$. The boundary orientation of $0 \times M \cong M$ is opposite to the original one and the boundary orientation of $1 \times M \cong M$ is the original one, if $I \times M$ carries the product orientation. We express these facts by the suggestive formula $\partial(I \times M) = 1 \times M - 0 \times M$. (These conventions suggest that homotopies should be defined with the cylinder $I \times X$.)

A diffeomorphism $f\colon M \to N$ between oriented manifolds ***respects the orientation*** if $T_x f$ is for each $x \in M$ orientation preserving. If M is connected, then f respects or reverses the orientation.

Problems

1. The atlas in 1.1.2 is not orienting; but S^n is orientable.
2. Show that a 1-manifold is orientable.
3. Let $f\colon M \to N$ be a smooth map and let A be the pre-image of a regular value $y \in N$. Suppose M is orientable, then A is orientable.

We specify an orientation as follows. Let M and N be oriented. We have an exact sequence $0 \to T_a A \xrightarrow{(1)} T_a M \xrightarrow{(2)} T_y N \to 0$, with inclusion (1) and differential $T_a f$ at

(2). This orients T_aA as follows: Let $v_1, \dots, v_k$ be a basis of T_aA, $w_1, \dots, w_l$ a basis of T_yN, and $u_1, \dots, u_l$ be pre-images in T_aM; then $v_1, \dots, v_k, u_1, \dots, u_l$ is required to be the given orientation of T_aM. These orientations induce an orientation of A. This orientation of A is called the ***pre-image orientation***.
4. Let $f \colon \mathbb{R}^n \to \mathbb{R}, (x_i) \mapsto \sum x_i^2$ and $S^{n-1} = f^{-1}(1)$. Then the pre-image orientation coincides with the boundary orientation with respect to $S^{n-1} \subset D^n$.

1.11 Tangent Bundle. Normal Bundle

The notions and concepts of bundle theory can now be adapted to the smooth category. A smooth bundle $p \colon E \to B$ has a smooth bundle projection p and the bundle charts are assumed to be smooth. A smooth subbundle of a smooth vector bundle has to be defined by smooth bundle charts. Let $\alpha \colon \xi_1 \to \xi_2$ be a smooth bundle morphism of constant rank; then $\operatorname{Ker} \alpha$ and $\operatorname{Im} \alpha$ are smooth subbundles. The proof of **??** can also be used in this situation. A smooth vector bundle has a smooth Riemannian metric; for the existence proof one uses a smooth partition of unity and proceeds as in **??**. Let ξ be a smooth subbundle of the smooth vector bundle η with Riemannian metric; then the orthogonal complement of ξ in η is a smooth subbundle.

Let M be a smooth n-manifold. Denote by TM the disjoint union of the tangent spaces $T_p(M)$, $p \in M$. We write a point of $T_p(M) \subset TM$ in the form (p, v) with $v \in T_p(M)$, for emphasis. We have the projection $\pi_M \colon TM \to M$, $(p, v) \mapsto p$. Each chart $k = (U, h, V)$ of M yields a bijection

$$\varphi_k \colon TU = \textstyle\bigcup_{p \in U} T_p(M) \to U \times \mathbb{R}^n, \quad (p, v) \mapsto (p, i_k(v)).$$

Here i_k is the morphism which is part of the definition of a tangent space. The map φ_k is a map over U and linear on fibres. The next theorem is a consequence of the general gluing procedure.

(1.11.1) Theorem. *There exists a unique structure of a smooth manifold on TM such that the $(TU, \varphi_k, U \times \mathbb{R}^n)$ are charts of the differential structure. The projection $\pi_M \colon TM \to M$ is then a smooth map, in fact a submersion. The vector space structure on the fibres of π_M give π_M the structure of an n-dimensional smooth real vector bundle with the φ_k as bundles charts.* □

The vector bundle $\pi_M \colon TM \to M$ is called the ***tangent bundle*** of M. A smooth map $f \colon M \to N$ induces a smooth fibrewise map $Tf \colon TM \to TN, (p, v) \mapsto (f(p), T_pf(v))$.

(1.11.2) Proposition. *Let $M \subset \mathbb{R}^q$ be a smooth n-dimensional submanifold. Then*

$$TM = \{(x, v) \mid x \in M,\, v \in T_xM\} \subset \mathbb{R}^q \times \mathbb{R}^q$$

is a $2n$-dimensional smooth submanifold.

Proof. Write M locally as $h^{-1}(0)$ with a smooth map $h\colon U \to \mathbb{R}^{q-n}$ of constant rank $q-n$ hat. Then TM is locally the pre-image of zero under

$$U \times \mathbb{R}^q \to \mathbb{R}^{q-n} \times \mathbb{R}^{q-n}, \quad (u,v) \mapsto (h(u), Dh(u)(v)),$$

and this map has constant rank $2(q-n)$; this can be seen by looking at the restrictions to $U \times 0$ and $u \times \mathbb{R}^q$. □

We can apply 1.11.2 to $S^n \subset \mathbb{R}^{n+1}$ and obtain the model of the tangent bundle of S^n, already used at other occasions.

Let $p\colon E \to B$ be a smooth vector bundle. Then E is a smooth manifold and we can ask for its tangent bundle.

(1.11.3) Proposition. *There exists a canonical exact sequence*

$$0 \to p^*E \xrightarrow{\alpha} TE \xrightarrow{\beta} p^*TM \to 0$$

of vector bundles.

Proof. The differential of p is a bundle morphism $Tp\colon TE \to TM$, and it induces a bundle morphism $\beta\colon TE \to p^*TM$ which is fibrewise surjective, since p is a submersion. We consider the total space of $p^*E \to E$ as $E \oplus E$ and the projection onto the first summand is the bundle projection. Let $(x,v) \in E_x \oplus E_x$. We define $\alpha(x,v)$ as derivative of the curve $t \mapsto x + tv$ at $t=0$. The bundle morphism α has an image contained in the kernel of β and is fibrewise injective. Thus, for reasons of dimension, the sequence is exact. □

Let the Lie group G act smoothly on M. We have an induced action

$$G \times TM \to TM, \quad (b,v) \mapsto (Tl_g)v.$$

This action is again smooth and the bundle projection is equivariant, i.e., $TM \to M$ is a smooth G-vector bundle.

(1.11.4) Proposition. *Let $\xi\colon E \to M$ be a smooth G-vector bundle. Suppose the action on M is free and proper. Then the orbit map $E/G \to M/G$ is a smooth vector bundle. We have an induced bundle map $\xi \to \xi/G$.*

The differential $Tp\colon TM \to T(M/G)$ of the orbit map p is a bundle morphism which factors over the orbit map $TM \to (TM)/G$ and induces a bundle morphism $q\colon (TM)/G \to T(M/G)$ over M/G. The map is fibrewise surjective. If G is discrete, then M and M/G have the same dimension, hence q is an isomorphism.

(1.11.5) Proposition. *For a free, proper, smooth action of the discrete group G on M we have a bundle isomorphism $(TM)/G \cong T(M/G)$ induced by the orbit map $M \to M/G$.* □

(1.11.6) Example. We have a bundle isomorphism $TS^n \oplus \varepsilon \cong (n+1)\varepsilon$. If $G = \mathbb{Z}/2$ acts on TS^n via the differential of the antipodal map and trivially on ε, then the said isomorphism transforms the action into $S^n \times \mathbb{R}^{n+1} \to S^n \times \mathbb{R}^{n+1}, (x,v) \mapsto (-x,-v)$. We pass to the orbit spaces and obtain an isomorphism $T(\mathbb{R}P^n) \oplus \varepsilon \cong (n+1)\eta$ with the tautological line bundle η over $\mathbb{R}P^n$. ◇

In the general case the map $q\colon (TM)/G \to T(M/G)$ has a kernel, a bundle $K \to M/G$ with fibre dimension $\dim G$.

(1.11.7) Example. The defining map $\mathbb{C}^{n+1} \smallsetminus 0 \to (\mathbb{C}^{n+1} \smallsetminus 0)/\mathbb{C}^* = \mathbb{C}P^n$ yields a surjective bundle map $q\colon (T(\mathbb{C}^{n+1} \smallsetminus 0))/\mathbb{C}^* \to T(\mathbb{C}P^n)$. The source of q is the $(n+1)$-fold Whitney sum $(n+1)\eta$ where $E(\eta)$ is the quotient of $(\mathbb{C}^{n+1} \smallsetminus 0) \times \mathbb{C}$ with respect to $(z,x) \sim (\lambda z, \lambda x)$ for $\lambda \in \mathbb{C}^*$ and $(z,x) \in (\mathbb{C}^{n+1} \smallsetminus 0) \times \mathbb{C}$. The kernel bundle of q is trivial: We have a canonical section of $(n+1)\eta$

$$\mathbb{C}P^n \to ((\mathbb{C}^{n+1} \smallsetminus 0) \times \mathbb{C}^{n+1})/\mathbb{C}^*, \quad [z] \mapsto (z,z)/\sim,$$

and the subbundle generated by this section is contained in the kernel of q. Hence the complex tangent bundle of $\mathbb{C}P^n$ satisfies $T(\mathbb{C}P^n) \oplus \varepsilon \cong (n+1)\eta$. ◇

Let $f\colon M \to N$ be an immersion. Then Tf is fibrewise injective. We pull back TN along f and obtain a fibrewise injective bundle morphism $i\colon TM \to f^*TN|M$. The quotient bundle is called the ***normal bundle*** of the immersion. In the case of a submanifold $M \subset N$ the normal bundle $\nu(M,N)$ of M in N is the quotient bundle of $TN|M$ by TM. If we give TN a smooth Riemannian metric, then we can take the orthogonal complement of TM as a model for the normal bundle. The normal bundle of $S^n \subset \mathbb{R}^{n+1}$ is the trivial bundle.

We will show that the total space of the normal bundle of an embedding $M \subset N$ describes a neighbourhood of M in N. We introduce some related terminology. Let $\nu\colon E(\nu) \to M$ denote the smooth normal bundle. A ***tubular map*** is a smooth map $t\colon E(\nu) \to N$ with the following properties:

(1) It is the inclusion $M \to N$ when restricted to the zero section.
(2) It embeds an open neighbourhood of the zero section onto an open neighbourhood U of M in N.
(3) The differential of t, restricted to $TE(\nu)|M$, is a bundle morphism

$$TE(\nu)|M \to TN|M.$$

We compose with the inclusion

$$E(\nu) \to E(\nu) \oplus TM \cong TE(\nu)|M$$

and the projection

$$TN|M \to E(\nu) = (TN|M)/TM.$$

We require that this composition is the identity.

The purpose of (3) is to exclude bundle automorphisms.

(1.11.8) Example. We restrict the exact sequence **??** to the zero section $i\colon M \subset E$. Then we obtain a canonical isomorphism $(\alpha, Ti)\colon E \oplus TM \cong TE|M$. The restriction of this isomorphism to E is a tubular map onto a tubular neighbourhood of $M \subset TE$. The normal bundle of the zero section $M \subset E$ equals E. ◇

(1.11.9) Remark (Shrinking). Let $t\colon E(\nu) \to N$ be a tubular map for a submanifold M. Then one can find by the process of shrinking another tubular map that embeds $E(\nu)$. There exists a smooth function $\varepsilon\colon M \to \mathbb{R}$ such that

$$E_\varepsilon(\nu) = \{y \in E(\nu)_x \mid \|y\| < \varepsilon(x)\} \subset U.$$

Let $\lambda_\eta(t) = \eta t \cdot (\eta^2 + t^2)^{-1/2}$. Then $\lambda_\eta\colon [0, \infty[\to [0, \eta[$ is a diffeomorphism with derivative 1 at $t = 0$. We obtain an embedding

$$h\colon E \to E, \quad y \mapsto \lambda_{\varepsilon(x)}(\|y\|) \cdot \|y\|^{-1} \cdot y, \quad y \in E(\nu)_x.$$

Then $g = fh$ is a tubular map that embeds $E(\nu)$. ◇

Let M be an m-dimensional smooth submanifold $M \subset \mathbb{R}^n$ of codimension k. We take $N(M) = \{(x, v) \mid x \in M, v \perp T_xM\} \subset M \times \mathbb{R}^n$ as the normal bundle of $M \subset \mathbb{R}^n$.

(1.11.10) Proposition. *$N(M)$ is a smooth submanifold of $M \times \mathbb{R}^n$, and the projection $N(M) \to M$ is a smooth vector bundle.*

Proof. Let $A\colon \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Its transpose A^t with respect to the standard inner product is defined by $\langle Av, w\rangle = \langle v, A^t w\rangle$. If A is surjective, then A^t is injective, and the relation image $(A^t) = (\text{kernel}\, A)^\perp$ holds; moreover $A \cdot A^t \in GL_k(\mathbb{R})$.

We define M locally as solution set: Suppose $U \subset \mathbb{R}^n$ is open, $\varphi\colon U \to \mathbb{R}^k$ a submersion, and $\varphi^{-1}(0) = U \cap M = W$. We set $N(M) \cap (W \times \mathbb{R}^n) = N(W)$. The smooth maps

$$\Phi\colon W \times \mathbb{R}^n \to W \times \mathbb{R}^k, \qquad (x, v) \mapsto (x, T_x\varphi(v))$$
$$\Psi\colon W \times \mathbb{R}^k \to W \times \mathbb{R}^n, \qquad (x, v) \mapsto (x, (T_x\varphi)^t(v))$$

satisfy

$$N(W) = \operatorname{Im}\Psi, \qquad T(W) = \operatorname{Ker}\Phi.$$

The composition $\Phi\Psi$ is a diffeomorphism: it has the form $(w,v) \mapsto (w, g_w(v))$ with a smooth map $W \to GL_k(\mathbb{R}), w \mapsto g_w$ and therefore $(w,v) \mapsto (w, g_w^{-1}(v))$ is a smooth inverse. Hence Ψ is a smooth embedding with image $N(W)$ and $\Psi^{-1}|N(W)$ is a smooth bundle chart. □

(1.11.11) Proposition. *The map $a\colon N(M) \to \mathbb{R}^n, (x,v) \mapsto x+v$ is a tubular map for $M \subset \mathbb{R}^n$.*

Proof. We show that a has bijective differential at each point $(x,0) \in N(M)$. Let $N_xM = T_xM^{\perp}$. Since $M \subset \mathbb{R}^n$ we consider T_xM as subspace of $\mathbb{R}^n$. Then $T_{(x,0)}N(M)$ is the subspace $T_xM \times N_xM \subset T_{(x,0)}(M \times \mathbb{R}^n) = T_xM \times \mathbb{R}^n$. The differential $T_{(x,0)}a$ is the identity on each of the subspaces T_xM and N_xM. Therefore we can consider this differential as the map $(u,v) \mapsto u+v$, i.e. essentially as the identity.

It is now a general topological fact **??** that a embeds an open neighbourhood of the zero section. Finally it is not difficult to verify property (3) of a tubular map. □

(1.11.12) Corollary. *If we transport the bundle map via the embedding a we obtain a smooth retraction $r\colon U \to M$ of an open neighbourhood U of $M \subset \mathbb{R}^n$.* □

(1.11.13) Theorem. *Let $f\colon X \to Y$ be a local homeomorphism. Let $A \subset X$ and $f\colon A \to f(A) = B$ be a homeomorphism. Suppose that each neighbourhood of B in Y contains a paracompact neighbourhood. Then there exists an open neighbourhood U of A in X which is mapped homeomorphically under f onto an open neighbourhood V of B in Y.*

Proof. Let $x \in X$ and $y = s(x)$. We choose open neighbourhoods U_x of y in U and V_x of x in N such that f induces a homeomorphism $U_x \to V_x$. The inverse is then a section s_x of f over V_x. Since $s(x) = s_x(x)$, both sections coincide on a neighbourhood of x in X.

We now choose a family $(V_j \mid j \in J)$ of open sets $V_j \subset N$ which cover X together with sectionss$_j\colon V_j \to U$ of f such that $s_j|V_j \cap X = s|V_j \cap X$. We can replace the V_j by smaller sets, if necessary, such that $(V_j \mid j \in J)$ is locally finite, e.g. if we choose a subordinate partition of unity (τ_j) and replace V_j with $\tau_j^{-1}]0,1]$.

Set $V = \cup_{j \in J} V_j$. There exists an open covering $(W_j \mid j \in J)$ of V such that $\overline{W}_j \subset V_j$ for $j \in J$. We could use for this purpose the sets $\overline{W}_j$ which are the supports of a partition of unity subordinate to (V_j). Let

$$W = \{x \in W \mid x \in \overline{W}_i \cap \overline{W}_j \Rightarrow s_i(x) = s_j(x)\}.$$

Then $X \subset W$. We define a continuous section s on W by the requirement that s is equal to s_j on $\overline{W}_j$; by construction, s is well-defined, and the continuity is seen by using local finiteness.

We show: W is a neighbourhood of X, and $s(W^\circ)$ is open. For this purpose we choose an open neighbourhood Q of $s(x)$, $x \in X$ which is mapped under f homeomorphically onto an open neighbourhood $f(Q) \subset V$ of x. This done, we choose an open neighbourhood A of x in V with the following properties:

(1) $A \subset f(Q)$.
(2) A meets only a finite number of $\overline{W}_j$, say $\overline{W}_{j(1)}, \ldots, \overline{W}_{j(k)}$.
(3) $x \in \overline{W}_{j(1)} \cap \ldots \cap \overline{W}_{j(k)}$.
(4) $A \subset V_{j(1)} \cap \ldots \cap V_{j(k)}$.
(5) $s_{j(t)}(A) \subset Q$, $1 \leq t \leq k$.

A choice of this type is possible: By local finiteness of $(\overline{W}_j)$ the set $\{j \in J \mid x \in \overline{W}_j\}$ is finite; let $\{j(1), \ldots, j(k)\}$ be this set. Suppose x is contained in

$$V_{j(1)}, \ldots, V_{j(k)}, V_{j(k+1)}, \ldots, V_j(l).$$

Then

$$A = f(Q) \cap V_{j(1)} \cap \ldots \cap V_{j(k)} \cap (V \smallsetminus \overline{W}_{j(k+1)}) \cap \ldots \cap (V \smallsetminus \overline{W}_{j(l)})$$

is an open neighbourhood of x in V with properties (1) – (4). By continuity of s_i, the pre-image $s_i^{-1}(Q)$ is an open neighbourhood of x. Thus we can shrink A such that $A \subset s_{j(t)}^{-1}(Q)$ holds.

Let $y \in A$. For $1 \leq a, b \leq k$ the equality $s_{j(a)}(y) = s_{j(b)}(y)$ holds, for by (5) both sets have the same image under f, and f is injective. Therefore A is contained in W. □

For embeddings of compact manifolds and their tubular maps one can apply another argument as in the following proposition.

(1.11.14) Proposition. *Let $\Phi\colon X \to Y$ be a continuous map of a locally compact space into a Hausdorff space. Let Φ be injective on the compact set $A \subset X$. Suppose that each $a \in A$ has a neighbourhood U_a in X on which Φ is injective. Then there exists a compact neighbourhood V of A in X on which Φ is an embedding.*

Proof. The coincidence set $K = \{(x, y) \in X \times X \mid \Phi(x) = \Phi(y)\}$ is closed in $X \times X$, since Y is a Hausdorff space. Let $D(B)$ be the diagonal of $B \subset X$. If Φ is injective on U_a, then $(U_a \times U_a) \cap K = D(U_a)$. Thus our assumptions imply that $D(X)$ is open in K and hence $W = X \times X \setminus (K \setminus D(X))$ open in $X \times X$. By assumption, $A \times A$ is contained in W. Since $A \times A$ is compact and X locally compact, there exists a compact neighbourhood V of A such that $V \times V \subset W$. Then $\Phi|V$ is injective and, being a map from a compact space into a Hausdorff space, an embedding. □

(1.11.15) Proposition. *A submanifold $M \subset N$ has a tubular map.*

Proof. We fix an embedding of $N \subset \mathbb{R}^n$. By **??** there exists an open neighbourhood W of V in $\mathbb{R}^n$ and a smooth retraction $r\colon W \to V$. The standard inner product on $\mathbb{R}^n$ induces a Riemannian metric on TN. We use as normal bundle for $M \subset N$ the model

$$E = \{(x, v) \in M \times \mathbb{R}^n \mid v \in (T_x M)^{\perp} \cap T_x N\}.$$

Again we use the map $f\colon E \to \mathbb{R}^n$, $(x, v) \mapsto x+v$ and set $U = f^{-1}(W)$. Then U is an open neighbourhood of the zero section of E. The map $g = rf\colon U \to N$ is the inclusion when restricted to the zero section. We claim that the differential of g at points of the zero section is the identity, if we use the identification $T_{(x,0)}E = T_x M \oplus E_x = T_x N$. On the summand $T_x M$ the differential $T_{(x,0)}g$ is obviously the inclusion $T_x M \subset T_x V$. For $(x, v) \in E_x$ the curve $t \mapsto (x, tv)$ in E has (x, v) as derivative at $t = 0$. Therefore we have to determine the derivative of $t \mapsto r(x + tv)$ at $t = 0$. The differential of r at $(x, 0)$ is the orthogonal projection $\mathbb{R}^n \to T_x N$, if we use the retraction r in **??**. The chain rule tells us that the derivative of $t \mapsto r(x+tv)$ at $t = 0$ is v. We now apply again **??**. One verifies property (3) of a tubular map. □

1.12 Embeddings

This section is devoted to the embedding theorem of Whitney:

(1.12.1) Theorem. *A smooth n-manifold has an embedding as a closed submanifold of* $\mathbb{R}^{2n+1}$.

We begin by showing that a compact n-manifold has an embedding into some Euclidean space. Let $f\colon M \to \mathbb{R}^t$ be a smooth map from an n-manifold M. Let $(U_j, \phi_j, U_3(0))$, $j \in \{1, \ldots, k\}$ be a finite number of charts of M. Choose a smooth function $\tau\colon \mathbb{R}^n \to [0, 1]$ such that $\tau(x) = 0$ for $\|x\| \geq 2$ and $\tau(x) = 1$ for $\|x\| \leq 1$. Define $\sigma_j\colon M \to \mathbb{R}$ by $\sigma_j(x) = 0$ for $x \notin U_j$ and by $\sigma_j(x) = \tau\phi_j(x)$ for $x \in U_j$; then σ_j is a smooth function on M. With the help of these functions we define

$$\Phi\colon M \to \mathbb{R}^t \times (\mathbb{R} \times \mathbb{R}^n) \times \cdots \times (\mathbb{R} \times \mathbb{R}^n) = \mathbb{R}^t \times \mathbb{R}^N$$

$$\Phi(x) = (f(x); \sigma_1(x), \sigma_1(x)\phi_1(x); \ldots; \sigma_k(x), \sigma_k(x)\phi_k(x)),$$

(k factors $\mathbb{R} \times \mathbb{R}^n$), where $\sigma_j(x)\phi_j(x)$ should be zero if $\phi_j(x)$ is not defined. The differential of this map has the rank n on $W_j = \phi_j^{-1}(U_1(0))$, because $\Phi(W_j) \subset V_j = \{(z; a_1, x_1; \ldots; a_k, x_k) \mid a_j \neq 0\}$, and the composition of $\Phi|W_j$ with $V_j \to \mathbb{R}^n, (z; a_1, x_1; \ldots) \mapsto a_j^{-1}x_j$ is ϕ_j. By construction, Φ is injective on $W = \bigcup_{j=1}^k W_j$, since $\Phi(a) = \Phi(b)$ implies $\sigma_j(a) = \sigma_j(b)$ for each j, and then $\phi_i(a) = \phi_i(b)$ holds for some i. Moreover, Φ is equal to f composed with $\mathbb{R}^t \subset \mathbb{R}^t \times \mathbb{R}^N$ on the complement of the $\phi_j^{-1}U_2(0)$. Hence if f is an (injective) immersion on the open set U, then Φ is an (injective) immersion on $U \cup W$. In particular, if M is compact, we can apply this argument to an arbitrary map f and $M = W$. Thus we have shown:

(1.12.2) Note. *A compact smooth manifold has a smooth embedding into some Euclidean space.* □

We now try to lower the embedding dimension by applying a suitable parallel projection.

Let $\mathbb{R}^{q-1} = \mathbb{R}^{q-1} \times 0 \subset \mathbb{R}^q$. For $v \in \mathbb{R}^q \setminus \mathbb{R}^{q-1}$ we consider the projection $p_v \colon \mathbb{R}^q \to \mathbb{R}^{q-1}$ with direction v, i.e., for $x = x_0 + \lambda v$ with $x_0 \in \mathbb{R}^{q-1}$ and $\lambda \in \mathbb{R}$ we set $p_v(x) = x_0$. In the sequel we only use vectors $v \in S^{q-1}$. Let $M \subset \mathbb{R}^q$. We remove the diagonal D and consider $\sigma \colon M \times M \setminus D \to S^{q-1}$, $(x, y) \mapsto N(x-y) = (x-y)/\|x-y\|$.

(1.12.3) Note. *$\varphi_v = p_v|M$ is injective if and only if v is not contained in the image of σ.*

Proof. The equality $\varphi_v(x) = \varphi_v(y), x \neq y$ and $x = x_0 + \lambda v$, $y = y_0 + \mu v$ imply $x - y = (\lambda - \mu)v \neq 0$, hence $v = \pm N(x-y)$. Note $\sigma(x,y) = -\sigma(y,x)$. □

Let now M be a smooth n-manifold in $\mathbb{R}^q$. We use the bundle of unit vectors

$$STM = \{(x, v) \mid v \in T_xM,\ \|v\| = 1\} \subset M \times S^{q-1}$$

and its projection to the second factor $\tau = \mathrm{pr}_2\,|STM \colon STM \to S^{q-1}$. The function $(x,v) \mapsto \|v\|^2$ on $TM \subset \mathbb{R}^q \times \mathbb{R}^q$ has 1 as regular value with pre-image STM, hence STM is a smooth submanifold of the tangent bundle TM.

(1.12.4) Note. *φ_v is an immersion if and only if v is not contained in the image of τ.*

Proof. The map φ_v is an immersion if for each $x \in M$ the kernel of $T_x p_v$ has trivial intersection with T_xM. The differential of p_v is again p_v. Hence $0 \neq z = p_v(z) + \lambda v \in T_xM$ is contained in the kernel of $T_x p_v$ if and only if $z = \lambda v$ and hence v is a unit vector in T_xM. □

(1.12.5) Theorem. *Let M be smooth compact n-manifold. Let $f \colon M \to \mathbb{R}^{2n+1}$ be a smooth map which is an embedding on a neighbourhood of a compact subset $A \subset M$. Then there exists for each $\varepsilon > 0$ an embedding $g \colon M \to \mathbb{R}^{2n+1}$ which coincides on A with f and satisfies $\|f(x) - g(x)\| < \varepsilon$ for $x \in M$.*

Proof. Suppose f embeds the open neighbourhood U of A and let $V \subset U$ be a compact neighbourhood of A. We apply the construction in the beginning of this section with chart domains U_j which are contained in $M \setminus V$ and such that the sets W_j cover $M \smallsetminus U$. Then Φ is an embedding on some neighbourhood of $M \setminus U$ and

$$\Phi\colon M \to \mathbb{R}^{2n+1} \oplus \mathbb{R}^N = \mathbb{R}^q, \quad x \mapsto (f(x), \Psi(x))$$

is an embedding which coincides on V with f (up to composition with the inclusion $\mathbb{R}^{2n+1} \subset \mathbb{R}^q$). For $2n < q-1$ the image of σ is nowhere dense and for $2n-1 < q-1$ the image of τ is nowhere dense (Sard). Therefore in each neighbourhood of $w \in S^{q-1}$ there exist vectors v such that $p_v \circ \Phi = \Phi_v$ is an injective immersion, hence an embedding since M is compact. By construction, Φ_v coincides on V with f. If necessary, we replace Ψ with $s\Psi$ (with small s) such that $\|f(x) - \Phi(x)\| \le \varepsilon/2$ holds. We can write f as composition of Φ with projections $\mathbb{R}^q \to \mathbb{R}^{q-1} \to \ldots \to \mathbb{R}^{2n+1}$ along the unit vectors $(0, \ldots, 1)$. Sufficiently small perturbations of these projections applied to Φ yield a map g such that $\|f(x) - g(x)\| < \varepsilon$, and, by the theorem of Sard, we find among these projections those for which g is an embedding. □

The preceding considerations show that that we need one dimension less for immersions.

(1.12.6) Theorem. *Let $f\colon M \to \mathbb{R}^{2n}$ be a smooth map from a compact n-manifold. Then there exists for each $\varepsilon > 0$ an immersion $h\colon M \to \mathbb{R}^{2n}$ such that $\|h(x) - f(x)\| < \varepsilon$ for $x \in M$. If $f\colon M \to \mathbb{R}^{2n+1}$ is a smooth embedding, then the vectors $v \in S^{2n}$ for which the projection $p_v \circ f\colon M \to \mathbb{R}^{2n}$ is an immersion are dense in S^{2n}.* □

Let $f\colon M \to \mathbb{R}$ be a smooth proper function from an n-manifold without boundary. Let $t \in \mathbb{R}$ be a regular value and set $A = f^{-1}(t)$. The manifold A is compact. There exists an open neighbourhood U of A in M and a smooth retraction $r\colon U \to A$.

(1.12.7) Proposition. *There exists an $\varepsilon > 0$ and open neighbourhood $V \subset U$ of A such that $(r, f)\colon V \to A \times\,]t-\varepsilon, t+\varepsilon[$ is a diffeomorphism.*

Proof. The map $(r, f)\colon U \to A \times \mathbb{R}$ has bijective differential at points of A. Hence there exists an open neighbourhood $W \subset U$ of A such that (r, f) embeds W onto an open neighbourhood of $A \times \{t\}$ in $A \times \mathbb{R}$. Since f is proper, each neighbourhood W of A contains a set of the form $V = f^{-1}\,]t-\varepsilon, t+\varepsilon[$. The restriction of (r, f) to V has the required properties. □

In a similar manner one shows that a proper submersion is locally trivial (Theorem of Ehresmann).

We now show that a non-compact n-manifold M has an embedding into $\mathbb{R}^{2n+1}$ as a closed subset. For this purpose we choose a proper smooth function $f\colon M \to \mathbb{R}_+$. We then choose a sequence $(t_k \mid k \in \mathbb{N})$ of regular values of f such that $t_k < t_{k+1}$ and $\lim_k t_k = \infty$. Let $A_k = f^{-1}(t_k)$ and $M_k = f^{-1}[t_k, t_{k+1}]$.

Choose $\varepsilon_k > 0$ small enough such that the intervals $J_k =]t_k - \varepsilon_k, t_k + \varepsilon_k[$ are disjoint and such that we have diffeomorphisms $f^{-1}(J_k) \cong A_k \times J_k$ of the type 1.12.7. We then use 1.12.7 in order to find embeddings $\Phi_k \colon f^{-1}(J_k) \to \mathbb{R}^{2n} \times J_k$ which have f as their second component. We then use the method of 1.12.5 to find an embedding $M_k \to \mathbb{R}^{2n} \times [t_k, t_{k+1}]$ which extends the embeddings Φ_k and Φ_{k+1} in a neighbourhood of $M_k + M_{k+1}$. All these embeddings fit together and yield an embedding of M as a closed subset of $\mathbb{R}^{2n+1}$.

(1.12.8) Proposition. *A smooth ∂-manifold M has a collar.*

Proof. There exists an open neighbourhood U of ∂M in M and a smooth retraction $r \colon U \to \partial M$. Choose a smooth function $f \colon M \to \mathbb{R}^+$ such that $f(\partial M) = \{0\}$ and the derivative of f at each point $x \in \partial M$ is non-zero. Then $(r, f) \colon U \to \partial M \times \mathbb{R}_+$ has bijective differential along ∂M. Therefore this map embeds an open neighbourhood V of ∂M onto an open neighbourhood W of $\partial M \times 0$. Now choose a smooth function $\varepsilon \colon \partial M \to \mathbb{R}_+$ such that $\{x\} \times [0, \varepsilon(x)[\subset W$ for each $x \in \partial M$. Then compose $\partial M \times [0, 1[\to \partial M \times \mathbb{R}_+, (x, s) \mapsto (x, \varepsilon(x)s)$ with the inverse of the diffeomorphism $V \to W$. □

(1.12.9) Theorem. *A compact smooth n-manifold B with boundary M has a smooth embedding of type* I *into D^{2n+1}.*

Proof. Let $j \colon M \to S^{2n}$ be an embedding. Choose a collar $k \colon M \times [0, 1[\to U$ onto the open neighbourhood U of M in B, and let $l = (l_1, l_2)$ be its inverse. We use the collar to extend j to $f \colon B \to D^{2n+1}$

$$f(x) = \begin{cases} \max(0, 1 - 2l_2(x)) j(l_1(x)) & x \in U \\ 0 & x \notin U. \end{cases}$$

Then f is a smooth embedding on $k(M \times [0, \frac{1}{2}[)$. As in the proof of 1.12.4 we approximate f by a smooth embedding $g \colon B \to D^{2n+1}$ which coincides with f on $k(M \times [0, \frac{1}{4}[)$ and which maps $B \smallsetminus M$ into the interior of D^{2n+1}. The image of g is then a submanifold of type I of D^{2n+1}. □

1.13 Approximation

Let M and N be smooth manifolds and $A \subset M$ a closed subset. We assume that $N \subset \mathbb{R}^p$ is a submanifold and we give N the metric induced by this embedding.

(1.13.1) Theorem. *Let $f \colon M \to N$ be continuous and $f|A$ smooth. Let $\delta \colon M \to]0, \infty[$ be continuous. Then there exists a smooth map $g \colon M \to N$ which coincides on A with f and satisfies $\|g(x) - f(x)\| < \delta(x)$ for $x \in M$.*

Proof. We start with the special case $N = \mathbb{R}$. The fact that f is smooth at $x \in A$ means, by definition, that there exists an open neighbourhood U_x of x and a smooth function $f_x \colon U_x \to \mathbb{R}$ which coincides on $U_x \cap A$ with f. Having chosen f_x, we shrink U_x, such that for $y \in U_x$ the inequality $\|f_x(y) - f(y)\| < \delta(y)$ holds.

Fix now $x \in M \setminus A$. We choose an open neighbourhood U_x of x in $M \setminus A$ such that for $y \in U_x$ the inequality $\|f(y) - f(x)\| < \delta(y)$ holds. We define $f_x \colon U_x \to \mathbb{R}$ in this case by $f_x(y) = (x)$.

Let $(\tau_x \mid x \in M)$ be a smooth partition of unity subordinate to $(U_x \mid x \in M)$. The function $g(y) = \sum_{x \in M} \tau_x(y) f_x(y)$ now has the required property.

From the case $N = \mathbb{R}$ one immediately obtains a similar result for $N = \mathbb{R}^p$. The general case will now be reduced to the special case $N = \mathbb{R}^p$. For this purpose we choose an open neighbourhood U of N in $\mathbb{R}^p$ together with a smooth retraction $r \colon U \to N$. We show in a moment:

(1.13.2) Lemma. *There exists a continuous function $\varepsilon \colon M \to]0, \infty[$ with the properties:*

(1) $U_x = U_{\varepsilon(x)}(f(x)) \subset U$ *for each* $x \in M$.

(2) *For each $x \in M$ the diameter of $r(U_x)$ is smaller than $\delta(x)$.*

Assuming this lemma, we apply 1.13.1 to $N = \mathbb{R}^p$ and ε instead of δ. This provides us with a map $g_1 \colon M \to \mathbb{R}^p$ which has an image contained in U. Then $g = r \circ g_1$ has the required properties. □

Proof. We first consider the situation locally. Let $x \in M$ be fixed. Choose $\gamma(x) > 0$ and a neighbourhood W_x of x such that $\delta(x) \geq 2\gamma(x)$ for $y \in W_x$. Let

$$V_x = r^{-1}(U_{\gamma(x)/2}(f(x)) \cap N).$$

The distance $\eta(x) = d(f(x), \mathbb{R}^p \setminus V_x)$ is greater that zero. We shrink W_x to a neighbourhood Z_x such that $\|f(x) - f(y)\| < \frac{1}{4}\eta(x)$ for $y \in Z_x$.

The function $f|Z_x$ satisfies the lemma with the constant function $\varepsilon = \varepsilon_x \colon y \mapsto \frac{1}{4}\eta(x)$. In order to see this, let $y \in Z_x$ and $\|z - f(y)\| < \frac{1}{4}\eta(x)$, i.e., $z \in U_y$. Then, by the triangle inequality, $\|z - f(x)\| < \frac{1}{2}\eta(x)$, and hence, by our choice of $\eta(x)$,

$$z \in V_x \subset U, \qquad r(z) \in U_{\gamma(x)/2}(f(x)).$$

If $z_1, z_2 \in U_y$, then the triangle inequality yields $\|r(z_1) - r(z_2)\| < \gamma(x) \leq \frac{1}{2}\delta(x)$. Therefore the diameter of $r(U_y)$ is smaller than $\delta(y)$.

After this local consideration we choose a partition of unity $(\tau_x \mid x \in M)$ subordinate to $(Z_x \mid x \in M)$. Then we define $\varepsilon \colon M \to]0, \infty[$ as $\varepsilon(x) = \sum_{a \in M} \frac{1}{4}\tau_a(x)\eta(a)$. This function has the required properties. □

(1.13.3) Proposition. *Let $f\colon M \to N$ be continuous. For each continuous map $\delta\colon M \to]0,\infty[$ there exists a continuous map $\varepsilon\colon M \to]0,\infty[$ with the following property: Each continuous map $g\colon M \to N$ with $\|g(x) - f(x)\| < \varepsilon(x)$ and $f|A = g|A$ is homotopic to f by a homotopy $F\colon M \times [0,1] \to N$ such that $F(a,t) = f(a)$ for $(a,t) \in A \times [0,1]$ and $\|F(x,t) - f(x)\| < \delta(x)$ for $(x,t) \in M \times [0,1]$.*

Proof. We choose $r\colon U \to N$ and $\varepsilon\colon M \to]0,\infty[$ as in 1.13.1 and 1.13.2. For $(x,t) \in M \times [0,1]$ we set $H(x,t) = t \cdot g(x) + (1-t) \cdot f(x) \in U_{\varepsilon(x)}(f(x))$. The composition is $F(x,t) = rH(x,t)$ is then a homotopy with the required properties. □

(1.13.4) Theorem. (1) *Let $f\colon M \to N$ be continuous and $f|A$ smooth. Then f is homotopic relative A to a smooth map. If f is proper and N closed in $\mathbb{R}^p$, then f is properly homotopic relative A to a smooth map.*

(2) *Let $f_0, f_1\colon M \to N$ be smooth maps. Let $f_t\colon M \to N$ be a homotopy which is smooth on $B = M \times [0,\varepsilon[\cup M \times]1-\varepsilon, 1] \cup A \times [0,1]$. Then there exists a smooth homotopy g_t from f_0 to f_1 which coincides on $A \times [0,1]$ with f. If f_t is a proper homotopy and N closed in $\mathbb{R}^p$, then g_t can be chosen as a proper homotopy.*

Proof. (1) We choose δ and ε according to 1.13.3 and apply 1.13.1. Then 1.13.3 yields a suitable homotopy. If f is proper, δ bounded, and if $\|g(x) - f(x)\| < \delta(x)$ holds, then g is proper.

(2) We now consider $M \times]0,1[$ instead of M and its intersection with B instead of A and proceed as in (1). □

1.14 Transversality

Let $f\colon A \to M$ and $g\colon B \to N$ be smooth maps. We form the pullback diagram

$$\begin{array}{ccc} C & \xrightarrow{F} & B \\ \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & M \end{array}$$

with $C = \{(a,b) \mid f(a) = g(b)\} \subset A \times B$. If $g\colon B \subset M$, then we identify C with $f^{-1}(B)$. If also $f\colon A \subset M$, then $f^{-1}(B) = A \cap B$. The space C can also be obtained as the pre-image of the diagonal of $M \times M$ under $f \times g$. The maps f and g are said to be ***transverse in*** $(a,b) \in C$ if

$$T_a f(T_a A) + T_b g(T_b B) = T_y M,$$

$y = f(a) = g(b)$. They are called ***transverse*** if this condition is satisfied for all points of C. If $g\colon B \subset M$ is the inclusion of a submanifold and $f(a) = b$, then we say that f is ***transverse*** to B in a if

$$T_a f(T_a M) + T_b B = T_b M$$

holds. If this holds for each $a \in f^{-1}(B)$, then f is called ***transverse*** to B. We also use this terminology if C is empty, i.e., we also call f and g transverse in this case. In the case that $\dim A + \dim B < \dim M$, the transversality condition cannot hold. Therefore f and g are then transverse if and only if C is empty. A submersion f is transverse to every g.

In the special case $B = \{b\}$ the map f is transverse to B if and only if b is a regular value of f. We reduce the general situation to this case.

We use a little linear algebra: Let $a\colon U \to V$ be a linear map and $W \subset V$ a linear subspace; then $a(U) + W = V$ if and only if the composition of a with the canonical projection $p\colon V \to V/W$ is surjective.

Let $B \subset M$ be a smooth submanifold. Let $b \in B$ and suppose $p\colon Y \to \mathbb{R}^k$ is a smooth map with regular value 0, defined on an open neighbourhood Y of b in M such that $B \cap Y = p^{-1}(0)$. Then:

(1.14.1) Note. *$f\colon A \to M$ is transverse to B in $a \in A$ if and only if a is a regular value of $p \circ f\colon f^{-1}(Y) \to Y \to \mathbb{R}^k$.*

Proof. The space $T_b B$ is the kernel of $T_b p$. The composition of $T_a f\colon T_a A \to T_b M / T_b B$ with the isomorphism $T_b M / T_b B \cong T_0 \mathbb{R}^k$ induced by $T_b\colon T_b M \to T_0 \mathbb{R}^k$ is $T_a(p \circ f)$. Now we apply the above remark from linear algebra. □

(1.14.2) Proposition. *Let $f\colon A \to M$ and $f|\partial A$ be smooth and transverse to the submanifold B of M of codimension k. Suppose B and M have empty boundary. Then $C = f^{-1}(B)$ is empty or a submanifold of type* I *of A of codimension k. The equality $T_a C = (T_a f)^{-1}(T_{f(a)} B)$ holds.* □

Let, in the situation of the last proposition, $\nu(C, A)$ and $\nu(B, M)$ the be normal bundles. Then Tf induces a smooth bundle map $\nu(C, A) \to \nu(B, M)$; for, by definition of transversality, $T_a f\colon T_a A / T_a C \to T_{f(a)} / T_{f(a)} B$ is surjective and then bijective for reasons of dimension.

From 1.14.1 we see that transversality is an "open condition": If $f\colon A \to M$ is transverse in a to B, then it is transverse in all points of a suitable neighbourhood of a, for a similar statement holds for regular points.

(1.14.3) Proposition. *Let $f\colon A \to M$ and $g\colon B \to M$ be smooth and let $y = f(a) = g(b)$. Then f and g are transverse in (a, b) if and only if $f \times g$ is transverse in (a, b) to the diagonal of $M \times M$.*

Proof. Let $U = T_a f(T_a A)$, $V = T_b g(T_b B)$, $W = T_y M$. The statement amounts to: $U + V = W$ and $(U \oplus V) + D(W) = W \oplus W$ are equivalent relations ($D(W)$ diagonal). By a small argument from linear one verifies this equivalence. □

(1.14.4) Corollary. *Suppose f and g are transverse. Then C is a smooth submanifold of $A \times B$. Let $c = (a,b) \in C$. We have a diagram*

$$\begin{array}{ccc} T_cC & \xrightarrow{TF} & T_bB \\ \downarrow{\scriptstyle TG} & & \downarrow{\scriptstyle Tg} \\ T_aA & \xrightarrow{Tf} & T_yM. \end{array}$$

It is bi-cartesian, i.e., $\langle Tf, Tg \rangle$ is surjective and the kernel is T_cC. Therefore the diagram induces an isomorphism of the cokernels of TG and Tg (and similarly of TF and Tf).

(1.14.5) Corollary. *Let a commutative diagram of smooth maps be given*

$$\begin{array}{ccccc} & & C & \xrightarrow{F} & B \\ & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{h} & A & \xrightarrow{f} & M. \end{array}$$

Let f be transverse to g and C as above. Then h is transverse to G if and only if fh is transverse to g.

Proof. The uses the isomorphisms of cokernels in 1.14.4. □

(1.14.6) Corollary. *We apply* 1.14.5 *to the diagram*

$$\begin{array}{ccccc} & & M & \longrightarrow & \{s\} \\ & & \downarrow{\scriptstyle i_s} & & \downarrow \\ W & \xrightarrow{f} & M \times S & \xrightarrow{\mathrm{pr}} & S \end{array}$$

and obtain: f is transverse to $i_s\colon x \mapsto (x,s)$ if and only if s is a regular value of $\mathrm{pr} \circ f$. □

Let $F\colon M \times S \to N$ be smooth and $Z \subset N$ a smooth submanifold. Suppose S, Z, and N have no boundary. For $s \in S$ we set $F_s\colon M \to N$, $x \mapsto F(x,s)$. We consider F as a parametrised family of maps F_s. Then:

(1.14.7) Theorem. *Suppose $F\colon M \times S \to N$ and $\partial F = F|(\partial M \times S)$ are transverse to Z. Then for almost all $s \in S$ the maps F_s and ∂F_s are both transverse to Z.*

Proof. By 1.14.2, $W = F^{-1}(Z)$ is a submanifold of $M \times S$ with boundary $\partial W = W \cap \partial(M \times S)$. Let $\pi\colon M \times S \to S$ be the projection. The theorem of Sard yields the claim if we can show: If $s \in S$ is a regular value of $\pi\colon W \to S$, then F_s is transverse to Z, and if $s \in S$ is regular value of $\partial\pi\colon \partial W \to S$, then ∂F_s is transverse to Z. But this follows from 1.14.6. □

(1.14.8) Theorem. *Let $f\colon M \to N$ be a smooth map and $Z \subset N$ a submanifold. Suppose Z and N have no boundary. Let $C \subset M$ be closed. Suppose f is transverse to Z in the points of C and ∂f transverse to Z in the points of $\partial M \cap C$. Then there exists a smooth map $g\colon M \to N$ which is homotopic to f, coincides on C with f and is on M and ∂M transverse to Z.*

Proof. We begin with the case $C = \emptyset$. We use the following facts: N is diffeomorphic to a submanifold of some $\mathbb{R}^k$; there exists an open neighbourhood U of N in $\mathbb{R}^k$ and a submersion $r\colon U \to N$ with $r|N = \mathrm{id}$. Let $S = E^k \subset \mathbb{R}^k$ be the open unit disk and set

$$F\colon M \times S \to N, \quad (x,s) \mapsto r(f(x) + \varepsilon(x)s).$$

Here $\varepsilon\colon M \to]0,\infty[$ is a smooth function for which this definition of F makes sense. We have $F(x,0) = f(x)$. We claim: F and ∂F are submersions. For the proof we consider for fixed x the map

$$S \to U_\varepsilon(f(x)), \quad s \mapsto f(x) + \varepsilon(x)s;$$

it is the restriction of an affine automorphism of $\mathbb{R}^k$ and hence a submersion. The composition with r is then a submersion too. Therefore F and ∂F are submersions, since already the restrictions to the $\{x\} \times S$ are submersions.

By 1.14.7, for almost all $s \in S$ the maps F_s and ∂F_s are transverse to Z. A homotopy from F_s to f is $M \times I \to N$, $(x,t) \mapsto F(x,st)$.

Let now C be arbitrary. There exists an open neighbourhood W of C in M such that f is transverse to Z on W and ∂f transverse to Z on $W \cap \partial M$. We choose a set V which satisfies $C \subset V^\circ \subset V \subset W^\circ$ and a smooth function $\tau\colon M \to [0,1]$ such that $M \setminus W \subset \tau^{-1}(1)$, $V \subset \tau^{-1}(0)$. Moreover we set $\sigma = \tau^2$. Then $T_x\sigma = 0$, whenever $\tau(x) = 0$. We now modify the map F from the first part of the proof

$$G\colon M \times S \to N, \quad (x,s) \mapsto F(x, \sigma(x)s)$$

and claim: G is transverse to Z. For the proof we choose a $(x,s) \in G^{-1}(Z)$. Suppose, to begin with, that $\sigma(x) \neq 0$. Then $S \to N$, $t \mapsto G(x,t)$ is, as a composition of a diffeomorphism $t \mapsto \sigma(x)t$ with the submersion $t \mapsto F(x,t)$, also a submersion and therefore G is regular at (x,s) and hence transverse to Z.

Suppose now that $\sigma(x) = 0$. We compute $T_{(x,s)}G$ at $(v,w) \in T_xM \times T_sS = T_xX \times \mathbb{R}^m$. Let

$$m\colon M \times S \to M \times S, \quad (x,s) \mapsto (x, \sigma(x)s).$$

Then

$$T_{(x,s)}m(v,w) = (v, \sigma(x)w + T_x\sigma(v)s).$$

The chain rule, applied to $G = F \circ m$, yields

$$T_{(x,s)}G(v,w) = T_{m(x,s)}F \circ T_{(x,s)}m(v,w) = T_{(x,0)}F(v,0) = T_x f(v),$$

since $\sigma(x) = 0$, $T_x\sigma = 0$ and $F(x,0) = f(x)$. Since $\sigma(x) = 0$, by choice of W and τ, f is transverse to Z in x, hence — since $T_{(x,s)}G$ and $T_x f$ have the same image — also G is transverse to Z in (x,s). A similar argument is applied to ∂G. Then one finishes the proof as in the case $C = \emptyset$. □

1.15 Gluing along Boundaries

We use collars in order to define a smooth structure if we glue manifolds with boundaries along pieces of the boundary. Another use of collars is the definition of a smooth structure on the product of two manifolds with boundary (smoothing of corners).

(1.15.1) Gluing along Boundaries. Let M_1 and M_2 be ∂-manifolds. Let $N_i \subset \partial M_i$ be a union of components of ∂M_i and let $\varphi\colon N_1 \to N_2$ be a diffeomorphism. We denote by $M = M_1 \cup_\varphi M_2$ the space which is obtained from $M_1 + M_2$ by the identification of $x \in N_1$ with $\varphi(x) \in N_2$. The image of M_i in M is again denoted by M_i. Then $M_i \subset M$ is closed and $M_i \smallsetminus N_i \subset M$ open. We define a smooth structure on M. For this purpose we choose collars $k_i\colon \mathbb{R}_- \times N_i \to M_i$ with open image $U_i \subset M_i$. The map

$$k\colon \mathbb{R} \times N_1 \to M, \quad (t,x) \mapsto \begin{cases} k_1(t,x) & t \le 0 \\ k_2(-t,\varphi(x)) & t \ge 0 \end{cases}$$

is an embedding with image $U = U_1 \cup_\varphi U_2$. We define a smooth structure (depending on k) by the requirement that $M_i \smallsetminus N_i \to M$ and k are smooth embeddings. This is possible since the structures agree on $(M_i \smallsetminus N_i) \cap U$. ◇

(1.15.2) Products. Let M_1 and M_2 be smooth ∂-manifolds. Then $M_1 \times M_2 \smallsetminus (\partial M_1 \times \partial M_2)$ has a canonical smooth structure by using products of charts for M_i as charts. We now choose collars $k_i\colon \mathbb{R}_- \times \partial M_i \to M_i$ and consider the composition λ

$$\begin{array}{ccc} \mathbb{R}^2_- \times \partial M_1 \times \partial M_2 & \xrightarrow{\pi \times \mathrm{id}} & \mathbb{R}_- \times \mathbb{R}_- \times \partial M_1 \times \partial M_2 \\ \downarrow \lambda & & \downarrow (1) \\ M_1 \times M_2 & \xleftarrow{k_1 \times k_2} & (\mathbb{R}_- \times \partial M_1) \times (\mathbb{R}_- \times \partial M_2). \end{array}$$

Here $\pi\colon \mathbb{R}^2_- \to \mathbb{R}^1_- \times \mathbb{R}^1_-$, $(r,\varphi) \mapsto (r, \frac{1}{2}\varphi + \frac{3\pi}{4})$, written in polar coordinates (r,φ), and (1) interchanges the 2. and 3. factor. There exists a unique smooth structure on $M_1 \times M_2$ such that $M_1 \times M_2 \smallsetminus (\partial M_1 \times \partial M_2) \subset M_1 \times M_2$ and λ are diffeomorphisms onto open parts of $M_1 \times M_2$. ◇

(1.15.3) Boundary Pieces. Let B and C be smooth n-manifolds with boundary. Let M be a smooth $(n-1)$-manifold with boundary and suppose that

$$\varphi_B \colon M \to \partial B, \qquad \varphi_C \colon M \to \partial C$$

are smooth embeddings. We identify in $B + C$ the points $\varphi_B(m)$ with $\varphi_C(m)$ for each $m \in M$. The result D carries a smooth structure with the following properties:

(1) $B \smallsetminus \varphi_B(M) \subset D$ is a smooth submanifold.
(2) $C \smallsetminus \varphi_C(M) \subset D$ is a smooth submanifold.
(3) $\iota \colon M \to D$, $m \mapsto \varphi_B(m) \sim \varphi_C(m)$ is a smooth embedding as a submanifold of type I.
(4) The boundary of D is diffeomorphic to the gluing of $\partial B \smallsetminus \varphi_B(M)^\circ$ with $\partial C \smallsetminus \varphi_C(M)^\circ$ via $\varphi_B(m) \sim \varphi_C(m)$, $m \in \partial M$.

The assertions (1) and (2) are understood with respect to the canonical embeddings $B \subset D \supset C$. We have to define charts about the points of $\iota(M)$, since the conditions (1) and (2) specify what happens about the remaining points. For points of $\iota(M \smallsetminus \partial M)$ we use collars of B and C and proceed as in 1.15.1. For $\iota(\partial M)$ we use the following device.

Choose collars $\kappa_B \colon \mathbb{R}_- \times \partial B \to B$ and $\kappa \colon \mathbb{R}_- \times \partial M \to M$ and an embedding $\tau_B \colon \mathbb{R} \times \partial M \to \partial B$ such that the next diagram commutes

$$\begin{array}{ccc} \mathbb{R} \times \partial M & \xrightarrow{\tau_B} & \partial B \\ \uparrow \cup & & \uparrow \varphi_B \\ \mathbb{R}_- \times \partial M & \xrightarrow{\kappa} & M. \end{array}$$

Here τ_B can essentially be considered as a tubular map, the normal bundle of $\varphi(\partial M)$ in ∂B is trivial. And κ is "half" of this normal bundle.

Then we form $\Phi_B = \kappa_B \circ (\mathrm{id} \times \tau_B) \colon \mathbb{R}_- \times \mathbb{R} \times \partial M \to B$. For C we choose in a similar manner κ_C and τ_C, but we require $\varphi_C \circ \kappa^- = \tau_C$ where $\kappa^-(m,t) = \kappa(m,-t)$. Then we define Φ_C from κ_C and τ_C. The smooth structure in a neighbourhood of $\iota(\partial M)$ is now defined by the requirement that $\alpha \colon \mathbb{R}_- \times \mathbb{R} \times \partial M \to D$ is a smooth embedding where

$$\alpha(r,\psi,m) = \begin{cases} \Phi_B(r, 2\psi - \pi/2, m), & \frac{\pi}{2} \le \psi \le \pi \\ \Phi_C(r, 2\psi - 3\pi/2, m), & \pi \le \psi \le \frac{3\pi}{2} \end{cases}$$

with the usual polar coordinates (r,ψ) in $\mathbb{R}_- \times \mathbb{R}$. ◇

(1.15.4) Connected Sum. Let M_1 and M_2 be n-manifolds. We choose smooth embeddings $s_i \colon D^n \to M_i$ into the interiors of the manifolds. In

$M_1 \smallsetminus s_1(E^n) + M_2 \smallsetminus s_2(E^n)$ we identify $s_1(x)$ with $s_2(x)$ for $x \in S^{n-1}$. The result is a smooth manifold (1.15.1). We call it the ***connected sum*** $M_1 \# M_2$ of M_1 and M_2. Suppose M_1, M_2 are oriented connected manifolds, assume that s_1 preserves the orientation and s_2 reverses it. Then $M_1 \# M_2$ carries an orientation such that the $M_i \smallsetminus s_i(E^n)$ are oriented submanifolds. One can show by isotopy theory that the oriented diffeomorphism type is in this case independent of the choice of the s_i. ◇

(1.15.5) Attaching Handles. Let M be an n-manifold with boundary. Let $s\colon S^{k-1} \times D^{n-k} \to \partial M$ be an embedding and identify in $M + D^k \times D^{n-k}$ the points $s(x)$ and x. The result carries a smooth structure (1.15.3) and is said to be obtained by attaching a k-handle to M.

Attaching a 0-handle is the disjoint sum with D^n. Attaching an n-handle means that a "hole" with boundary S^{n-1} is closed by inserting a disk. A fundamental result asserts that each (smooth) manifold can be obtained by successive attaching of handles. A proof uses the so-called Morse-theory (see e.g. [**?**] [**?**]). A handle decomposition of a manifold replaces a cellular decomposition, the advantage is that the handles are themselves n-dimensional manifolds. ◇

(1.15.6) Elementary Surgery. If M' arises from M by attaching a k-handle, then $\partial M'$ is obtained from ∂M by a process called ***elementary surgery***. Let $h\colon S^{k-1} \times D^{n-k} \to X$ be an embedding into an $(n-1)$-manifold with image U. Then $X \smallsetminus U^\circ$ has a piece of the boundary which is via h diffeomorphic to $S^{k-1} \times S^{n-k-1}$. We glue the boundary of $D^k \times S^{n-k-1}$ with h; in symbols

$$X' = (X \smallsetminus U^\circ) \cup_h D^k \times S^{n-k-1}.$$

The transition from X to X' is called ***elementary surgery of index*** k at X via h. The method of surgery is very useful for the construction of manifolds with prescribed topological properties. See [**?**] [**?**] to get an impression of surgery theory. ◇

Problems

1. The subsets of $S^{m+n+1} \subset \mathbb{R}^{m+1} \times \mathbb{R}^{n+1}$

$$D_1 = \{(x,y) \mid \|x\|^2 \geq \tfrac{1}{2}, \|y\|^2 \leq \tfrac{1}{2}\}, \quad D_2 = \{(x,y) \mid \|x\|^2 \leq \tfrac{1}{2}, \|y\|^2 \geq \tfrac{1}{2}\}$$

are diffeomorphic to $D_1 \cong S^m \times D^{n+1}$, $D_2 \cong D^{m+1} \times S^n$. They are smooth submanifolds with boundary of S^{m+n+1}. Hence S^{m+n+1} can be obtained from $S^m \times D^{n+1}$ and $D^{m+1} \times S^n$ by identifying the common boundary $S^m \times S^n$ with the identity. A diffeomorphism $D_1 \to S^m \times D^{n+1}$ is $(z,w) \mapsto (\|z\|^{-1}z, \sqrt{2}w)$.

2. Let M be a manifold with non-empty boundary. Identify two copies along the boundary with the identity. The result is the ***double*** $D(M)$ of M. Show that $D(M)$ for a compact M is the boundary of some compact manifold. (Hint: Rotate M about ∂M about 180 degrees.)

3. Show $M \# S^n \cong M$ for each n-manifold M.

4. Study the classification of closed connected surfaces. The orientable surfaces are S^2 and connected sums of tori $T = S^1 \times S^1$. The non-orientable ones are connected sums of projective plains $P = \mathbb{R}P^2$. The relation $T \# P = P \# P \# P$ holds. Connected sum with T is classically also called attaching of a handle.

Chapter 2

Manifolds II

2.1 Vector Fields and their Flows

A vector field is defined geometrically as a section of the tangent bundle. It also has an interpretation as a linear differential operator of first order. If $U \subset \mathbb{R}^n$ is open, we write as usual $T(U) = U \times \mathbb{R}^n$, and then a vector field on U is determined by its second component $X\colon U \to \mathbb{R}^n$. The vector field is smooth, if this map is smooth. Let $f\colon U \to \mathbb{R}$ be smooth and let $X(p) = \sum_{i=1}^n a_i(p)e_i$ be a linear combination of the standard basis. Then $(T_pf)(X(p)) \in T_{f(p)}\mathbb{R} = \mathbb{R}$ equals $\sum_{i=1}^n a_i(p)\partial f/\partial x_i(p)$. The vector $X(p)$ is determined by its action on smooth functions. For this reason we denote $X(p)$ as differential operator $\sum_{i=1}^n a_i(p)\partial/\partial x_i$, and instead of $(T_pf)(X(p))$ we write $X(p)f$, in order to indicate that $X(p)$ is an operator on smooth functions. If X is smooth, the smooth function $p \mapsto X(p)f$ is denoted Xf. We use a similar notation for smooth vector fields X on smooth manifolds M and for smooth functions $f\colon M \to \mathbb{R}$.

Let $X\colon M \to TM$ be a smooth vector field on M. An ***integral curve*** of X with ***initial condition*** $p \in M$ is a smooth map $\alpha\colon J \to M$ from an open interval $0 \in J \subset \mathbb{R}$ such that $\alpha(0) = p$ and $\alpha'(t) = X(\alpha(t))$ for each $t \in J$. We also say: α starts in p. We have used the notation $\alpha'(t)$ for the velocity vector of α. The next theorem globalizes some standard results about linear differential equations (existence, uniqueness, dependence on initial conditions). Let $f\colon M \to N$ be a diffeomorphism and $X\colon M \to TM$ a smooth vector field. Then $Y = Tf \circ X \circ f^{-1}\colon N \to TN$ is a smooth vector field on N. Let $\alpha\colon J \to M$ be an integral curve of X. We have the constant vector field $\partial/\partial t$ on J. We obtain a commutative diagram

$$\begin{array}{ccccc} TJ & \xrightarrow{T\alpha} & TM & \xrightarrow{Tf} & TN \\ \uparrow{\scriptstyle \frac{\partial}{\partial t}} & & \uparrow{\scriptstyle X} & & \uparrow{\scriptstyle Y} \\ J & \xrightarrow{\alpha} & M & \xrightarrow{f} & N. \end{array}$$

The left square commutes, because α is an integral curve, the right square commutes by definition of Y. Hence the composition $f \circ \alpha$ is an integral curve of Y.

(2.1.1) Theorem. *Let X be a smooth vector field on M. There exists an open set $D(X) \subset \mathbb{R} \times M$ and a smooth map $\Phi\colon D(X) \to M$ such that:*

(1) $0 \times M \subset D(X)$.

(2) $t \mapsto \Phi(t,p)$ *is an integral curve of X with initial condition p.*

(3) *If $\alpha\colon J \to M$ is an integral curve with initial condition p, then J is contained in $D(X) \cap (\mathbb{R} \times p) =]a_p, b_p[$, and $\alpha(t) = \Phi(t,p)$ holds for $t \in J$.*

(4) *The relations $\Phi(0,x) = x$ and $\Phi(s,\Phi(t,x)) = \Phi(s+t,x)$ hold whenever the left side is defined (then the right side is also defined). In particular $]a_p - t, b_p - t[=]a_{\Phi(t,p)}, b_{\Phi(t,p)}[$ holds for each $t \in]a_p, b_p[$.*

(5) *If M is compact or, more generally, X has compact support, then $D(X) = \mathbb{R} \times M$.* □

The map Φ in 2.1.1 is called the ***flow*** of the vector field X. If $D(X) = \mathbb{R} \times M$, then we say that X is ***globally integrable***. In that case the flow $\Phi\colon \mathbb{R} \times M \to M$ has the properties $\Phi(0,x) = x$ and $\Phi(s,\Phi(t,x)) = \Phi(s+t,x)$; it is a smooth action of the additive group $\mathbb{R}$ on M.

An integral curve with finite interval of definition leaves each compact set. More precisely:

(2.1.2) Proposition. *Let $\alpha\colon]a,b[\to M$ be an integral curve with maximal interval of definition, let $K \subset M$ be compact, and $b < \infty$. Then there exists $c < b$ such that $\alpha(t) \notin K$ for $t > c$.* □

(2.1.3) Theorem. *Integral curves have one of the following types:*

(1) *The curve is constant. (The initial condition is a zero of the vector field.)*

(2) *The curve is an injective immersion.*

(3) *The curve α is periodic and defined on $\mathbb{R}$, i.e., there exists $\tau > 0$ such that $\alpha(s) = \alpha(t)$ if and only if $s - t \in \tau\mathbb{Z}$. The image of α is then a compact submanifold of M, and α induces a diffeomorphism $\exp(2\pi i t) \mapsto \alpha(\tau t)$ of S^1 with the image of α.* □

(2.1.4) Proposition. *Let M be a smooth submanifold of N and $A \subset M$ a subset which is closed in N. Given a smooth vector field X on M, there exists* *a smooth vector field Y on N such that $Y|A = X|A$ and Y is zero in the complement of a neighbourhood of A in N.*

Proof. Using adapted charts we see that X has about each point of M an extension to a neighbourhood of the point in N. Let $(U_j \mid j \in J)$ be an open covering of N. Choose an extension Y^j of $X|N \cap U_j$ to U_j. Let (τ_j) be a smooth partition of unity subordinate to (U_j). Then $Y_p = \sum_j \tau_j Y^j$ is an extension of X. We apply this construction to a covering which consists of $N \smallsetminus A$ and an open covering of A. □

(2.1.5) Proposition. *Let M be a ∂-manifold. There exist smooth vector fields X on M such that for each $p \in \partial M$ the vector X_p points inwards.*

Proof. From the definition of a ∂-manifold on sees immediately that each $p \in \partial M$ has an open neighbourhood $U(p)$ in M and a smooth vector field $X(p)$ on $U(p)$ which points inwards along $U(p) \cap \partial M$. Choose a smooth partition of unity (τ_p) subordinate to the covering $(U(p) \mid p \in \partial M)$, $U(0) = M \smallsetminus \partial M$. Then $X = \sum_p \tau_p X(p)$ has the required properties. □

Let M be a smooth ∂-manifold. A ***collar*** for M is a smooth map

$$k\colon \mathbb{R}_- \times \partial M \to M$$

which is a diffeomorphism onto an open neighbourhood of ∂M in M and satisfies $k(x,0) = x$ for $x \in \partial M$. The next theorem will be proved later.

(2.1.6) Theorem. *A smooth ∂-manifold has a collar.*

Proof. Let X be a smooth vector field on M which points inwards along ∂M. Through each point $x \in \partial M$ passes a maximal integral curve $k_x\colon [0, b_x[\, \to M$ which begins in x. The set $d(X) = \{(x,t) \mid t \in [0, b_x[\,\}$ is open in $\partial M \times [0, \infty[$ and $\kappa\colon d(X) \to M$, $(x,t) \mapsto k_x(t)$ is smooth. The map κ has maximal rank in the points of $\partial M \times 0$. Hence there exists an open neighbourhood U of $\partial M \times 0$ in $d(X)$ which is mapped diffeomorphically under κ onto an open neighbourhood V of ∂M in M.

There exists a positive smooth function $\varepsilon\colon \partial M \to \mathbb{R}$ such that $\{(x,t) \mid 0 \le t < \varepsilon(x)\} \subset U$. The map $k(x,t) = \kappa(x, \varepsilon(x)t)$ is then a collar. □

Problems

1. A smooth flow on M consists of an open neighbourhood $\mathcal{O}$ of $0 \times M$ in $\mathbb{R} \times M$ and a smooth map $\Psi\colon \mathcal{O} \to M$ such that:

(1) $\{t \in \mathbb{R} \mid (t,x) \in \mathcal{O}\}$ is an open interval $]a_x, b_x[$.

(2) For each $t \in\,]a_x, b_x[$ the equality $]a_x - t, b_x - t[\, = \,]a_{\Psi(t,x)}, b_{\Psi(t,x)}[$ holds.

(3) $\Psi(0,x) = x$; and $\Psi(s,\Psi(t,x)) = \Psi(s+t,x)$ for each $t \in]a_x, b_x[$ and $s+t \in]a_x, b_x[$.

The flow line through $x \in M$ is the curve $\alpha_x \colon]a_x, b_x[\to M$, $t \mapsto \Psi(t,x)$. Let $X(x) = \alpha_x'(0)$. This is a smooth vector field X on M, and α_x an integral curve which starts in x. The flow Φ 2.1.1 of X extends Ψ to a possibly larger set of definition. In this sense, smooth vector fields correspond to maximal smooth flows.

2. If A is a closed submanifold of M and X a vector field on M such that $X|A$ is a vector field on A (i.e. $X(a) \in T_a(A) \subset T_a(M)$ for $a \in A$), then an integral curve which starts in A stays inside A. If $X|A$ is globally integrable, then it is not necessary to assume that A is closed.

2.2 Proper Submersions

(2.2.1) Theorem. *Let $f \colon M \to J$ be a proper submersion onto an open interval $J \subset \mathbb{R}$. Then there exists a diffeomorphism $\Phi \colon Q \times J \to M$ such that $f \circ \Phi$ is the projection onto J.*

Proof. There exists a smooth vector field X on M such that $T_p f(X_p) = X_p f = 1$ for each $p \in M$. (Proof: choose a Riemannian metric on M, form the gradient field of f and divide it at each point by the square of its norm.) Let Ψ be the flow of X. We fix $\sigma \in J$ and set $Q = f^{-1}(\sigma)$. Let $x \in Q$ and $\alpha \colon I \to M$ the maximal integral curve of X starting at $\alpha(\sigma) = x$. Since $f \circ \alpha$ has derivative 1, we conclude $f\alpha(t) = t$. Since f is proper and an integral curve with finite interval of definition leaves each compact set, we see that $I = J$. We can write α in the form $\alpha(t) = \Psi(t-\sigma, x)$. From the relation $f\Psi(t,x) = f(x) + t$ we conclude that Ψ is defined on $\{(t,x) \mid t - f(x) \in J\} \subset \mathbb{R} \times M$. The smooth map

$$\Phi \colon Q \times J \to M, \quad (x,t) \mapsto \Psi(t-\sigma, x)$$

satisfies $f \circ \Phi = \mathrm{pr}_J$. In order to see that Φ is a diffeomorphism it suffices to show that for each $t \in J$ the fibre $\mathrm{pr}_J^{-1}(t) = Q \times \{t\} \cong Q = f^{-1}(\sigma)$ is mapped by a diffeomorphism onto $f^{-1}(t)$. The map in question is

$$x \in f^{-1}(\sigma) \mapsto \Psi(t-\sigma, x) \in f^{-1}(t)$$

It has the inverse $y \mapsto \Psi(\sigma - t, y)$, by one of the basic properties of a flow. □

We have used the properness of f to determine the maximal interval of definition of an integral curve. There exist non-proper submersions with compact fibres.

(2.2.2) Proposition. *Let $f \colon M \to N$ be a submersion. Suppose the fibres $f^{-1}(y)$ are compact and connected. Then f is proper.*

Proof. We fix $y_0 \in N$. Since $f^{-1}(y_0)$ is compact, this set has a compact neighbourhood V. We show: There exists a compact neighbourhood W of y_0 such that $f^{-1}(W)$ is contained in V. Suppose that for each compact neighbourhood A of y_0 the intersection with the boundary $f^{-1}(A) \cap \mathrm{Bd}(V) \neq \emptyset$. Since

$$\bigcap_A f^{-1}(A) \cap \mathrm{Bd}(V) = f^{-1}(y_0) \cap \mathrm{Bd}(V) = \emptyset,$$

by compactness of $\mathrm{Rd}(V)$ the intersection of a finite number of sets $f^{-1}(A) \cap \mathrm{Bd}(V)$ is empty, and this is impossible. Hence there exists A_0 such that $f^{-1}(A_0) \cap \mathrm{Bd}(V) = \emptyset$. We now use the connectedness of $f^{-1}(a)$ and conclude from $f^{-1}(a) \cap \mathrm{Rd}(V) = \emptyset$ that $f^{-1}(a)$ is either contained in V° or in $M \smallsetminus V$. Since f is a submersion, $f(V)$ is a neighbourhood of y_0, and $f^{-1}(a) \cap V \neq \emptyset$ for $a \in f(V)$. Hence $W = A_0 \cap f(V)$ is a suitable neighbourhood. The closed subset $f^{-1}(W)$ of V is compact. Therefore each point of N has a compact neighbourhood with compact pre-image. This implies that each compact subset of N has a compact pre-image. □

(2.2.3) Theorem. *Let $f\colon M \to U$ be a proper submersion onto the product $U = \prod_{j=1}^k J_j \subset \mathbb{R}^k$ of open intervals $J_j \subset \mathbb{R}$. Then there exists a diffeomorphism $\Phi\colon Q \times U \to M$ such that $f \circ \Phi$ is the projection onto U.*

Proof. Induction over k. The case $k = 1$ is settled by 2.2.1. Let f_j be the die j-th component of f. We choose a smooth vector field X_1 on M which is mapped under Tf_1 to $\partial/\partial t_1$ and under Tf_j, $j > 1$, to zero. (Existence: Choose a Riemannian metric; form the orthogonal complement of the kernel bundle of Tf; then Tf is an isomorphism on the fibres of this complement; the values of the vector field X_1 should be taken in this complement; all these conditions determine the vector field uniquely.) The integral curve α of X_1 through $x \in f^{-1}(\sigma_1, \dots, \sigma_k)$ origin $\alpha(\sigma_1) = x$ has J_1 as maximal interval of definition. Let $Q_1 = f_1^{-1}(\sigma_1)$. As in the proof of 2.2.1 we obtain from the flow of X_1 a diffeomorphism $\Phi_1\colon Q_1 \times J_1 \to M$ such that $f_1 \circ \Phi_1$ is the projection onto J_1. We use the induction hypothesis for $(f_2, \dots, f_k)|Q_1$. □

(2.2.4) Corollary (Ehresmann). *A proper submersion $f\colon M \to N$ is locally trivial.* □

We now prove the Whitney embedding theorem 1.12.1 for not necessarily compact manifolds. Let $h\colon M \to [0, \infty[$ be a smooth proper function. We set $U_i = h^{-1}]i - \frac{1}{4}, i + \frac{5}{4}[$ and $K_i = h^{-1}]i - \frac{1}{3}, i + \frac{4}{3}[$. Then U_i is open, K_i compact and $\overline{U}_i \subset K_i$. By the procedure of 1.12.4 there exist smooth maps $s_i\colon M \to \mathbb{R}^{2n+1}$ which embed a neighbourhood of $\overline{U}_i$ and which are zero away from K_i. If necessary, we compose with a suitable diffeomorphism of $\mathbb{R}^{2n+1}$ and assume that the s_i have an image contained in $D = D^{2n+1}$. We define f_j as the sum of the s_i with $i \equiv j \bmod 2$ and $f = (f_0, f_1, h)\colon M \to \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R} = V$. By construction $f(M) \subset D \times D \times \mathbb{R} = K \times \mathbb{R}$. Let $f(x) = f(y)$; then

$h(x) = h(y)$; therefore there exists i with $x, y \in U_i$; since then s_i is injective, we conclude that $x = y$. Hence f is an embedding, and the image is closed, since f is proper. Again by the procedure of 1.12.4 there exists a projection $p\colon V \to H$ onto a subspace H of dimension $2n+1$ such that $p \circ f$ is an injective immersion. Moreover we can choose p in such a manner that the kernel of p has trivial intersection with the kernel of the projection $q\colon V \to \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}$. We claim that $p \circ f$ is proper. Suppose that $C \subset H$ is compact. Then $(p \circ f)^{-1}(C) = f^{-1}p^{-1}(C) \subset (K \times \mathbb{R}) \cap p^{-1}(C) = (q,p)^{-1}(K \times C)$ is compact, since the linear inclusion (q,p) is proper.

2.3 Isotopies

Let $h_0, h_1\colon M \to N$ be smooth embeddings. An ***isotopy*** from h_0 to h_1 is a smooth map $H\colon M \times \mathbb{R} \to N$ with the properties:

(1) $h_0(x) = H(x,0)$, $h_1(x) = H(x,1)$ for $x \in M$.

(2) $H_t\colon M \to N$, $x \mapsto H(x,t)$ is a smooth embedding.

(3) There exists $\varepsilon \in]0,1[$ such that $H_t = H_0$ for $t < \varepsilon$ and $H_t = H_1$ for $t > 1 - \varepsilon$.

We call h_0 and h_1 ***isotopic*** if there exists an isotopy from h_0 to h_1. This relation is an equivalence relation; the proof is the same as for the homotopy relation. We have arranged the definition so that the product isotopy (defined as the product homotopy) is smooth.

In analogy to the homotopy notion one would expect to define an isotopy from h_0 to h_1 as a smooth map $h\colon M \times [0,1] \to N$ which yields for each $t \in I$ an embedding h_t. If h is a smooth map of this type and $\varphi\colon \mathbb{R} \to [0,1]$ a smooth function, $\varphi(t) = 0$ for $t < \varepsilon$ and $\varphi(t) = 1$ for $t > 1 - \varepsilon$, then $(x,t) \mapsto h(x, \varphi(t))$ is an isotopy in the sense of our definition, i.e., satisfying also condition (3). We use this device without further mentioning.

A ***diffeotopy*** of the smooth manifold N is a smooth map $D\colon N \times \mathbb{R} \to N$ such that $D_0 = \mathrm{id}(N)$ and D_t is a diffeomorphism for each t.

Let $h_0, h_1\colon M \to N$ be smooth embeddings. A diffeotopy D of N is said to be an ***ambient isotopy*** from h_0 to h_1 if

$$h\colon M \times \mathbb{R} \to N, \quad (x,t) \mapsto D(h_0(x), t) = D_t(h_0(x))$$

is an isotopy from h_0 to h_1. We say is this case that the isotopy is ***carried along*** by the diffeotopy. Only the restriction $D|N \times [0,1]$ matters. The relation “ambient isotopic” is an equivalence relation on the set of embeddings.

(2.3.1) Example. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a smooth embedding which preserves the origin. Then f is isotopic to the differential $Df(0)$. We write $f(x) =$

$\sum_{i=1}^n x_i g_i(x)$ with smooth functions $g_i\colon \mathbb{R}^n \to \mathbb{R}^n$. Note that $Df(0)\colon v \mapsto \sum_{i=1}^n v_i g_i(0)$. An isotopy is now defined by

$$(x,t) \mapsto \sum_{i=1}^n x_i g_i(tx) = \begin{cases} t^{-1}f(tx) & t > 0 \\ Df(0) & t = 0. \end{cases}$$

If h_0 and h_1 are ambient isotopic and if $h_0(M) \subset N$ is closed, then also $h_1(M) \subset N$ is closed. This shows that not every isotopy just constructed can be extended to an ambient isotopy. ◇

(2.3.2) Theorem. *Any two strong tubular maps are isotopic as strong tubular maps.*

Proof. (1) The proof generalizes the method of the previous example. Let M be a smooth m-submanifold of the smooth n-manifold N. Let f_0 and f_1 be tubular maps. We assume, to begin with, an additional hypothesis: For each $p \in M$ there exist chart domains U and V of M about p such that $E(\nu)$ is trivial over U and V and such that $f_0(E(\nu)|U) \subset f_1(E(\nu)|V)$ holds. Then f_0 and f_1 are strongly isotopic.

The hypothesis $f_0(E(\nu)) \subset f_1(E(\nu))$ allows us to consider

$$\varphi = f_1^{-1} f_0\colon E(\nu) \to E(\nu);$$

and we show that φ is strongly isotopic to the identity. The isotopy ψ_t, $t \in [0,1]$ is defined for $t > 0$ as $\psi_t(v) = t^{-1}\varphi(tv)$, and ψ_0 is the identity. For each t the map ψ_t is a smooth embedding onto an open neighbourhood of ?? which is the identity on the zero section. We show that ψ_t is strong and that ψ is smooth. By our hypothesis, we can express φ in suitable local coordinates in the form

$$\varphi\colon U \times \mathbb{R}^{n-m} \to V \times \mathbb{R}^{n-m}, \quad (x,y) \mapsto (f(x,y), g(x,y))$$

with $f(x,0) = x$, $g(x,0) = 0$. There exists a presentation

$$g(x,y) = \sum_{i=1}^{n-m} y_i g_i(x,y)$$

with smooth functions $g_i\colon \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ which satisfy $g_i(x,0) = \frac{\partial g}{\partial y_i}(x,0)$. This shows us that

$$(x,y,t) \mapsto \psi_t(x,y) = (f(x,ty), \textstyle\sum_i y_i g_i(x,ty))$$

is smooth in x, y, t. The derivative at the zero section satisfies

$$\frac{\partial \psi_t}{\partial y_i}(x,0) = g_i(x,0).$$

Since f_0 and f_1 are strong, the matrix with rows $g_i(x,0)$ is the unit matrix. Hence ψ_t is strong.

(2) We now verify that we can arrange for the additional assumption of part (1) of the proof. Let $E(\nu)$ be trivial over the chart domain V about p. Then $f_1(E(\nu)|V)$ is an open neighbourhood of $V \subset M \subset N$ in N. Let $W \subset E(\nu)$ be an open neighbourhood of $p \in M$ such that $f_0(W) \subset f_1(E(\nu)|V)$. We choose in W a set of the form $E(\nu,\eta)|U$, $\eta > 0$ with chart domain U over which $E(\nu)$ is trivial. There exists a smooth positive function $\varepsilon\colon M \to \mathbb{R}$ such that for each (U,η) the inequalities $\varepsilon(x) < \eta$, $x \in U$ hold. From $E(\nu,\varepsilon)$ we obtain a suitable tubular map by shrinking f_0. The argument in (1) shows that shrinking does not change the strong isotopy class. □

We construct diffeotopies by integrating suitable vector fields. For this purpose we consider isotopies as “movie”. Let $h\colon M \times \mathbb{R} \to N$ be a smooth map. The ***movie*** of h is the smooth map

$$h^{\#}\colon M \times \mathbb{R} \to N \times \mathbb{R}, \quad (x,t) \mapsto (h_t(x), t).$$

Since $h^{\#}(M \times t) \subset N \times t$, we call $h^{\#}$ ***height preserving***. If h is an isotopy, then $h^{\#}$ is a height preserving immersion. We call h ***strict*** if $h^{\#}$ is an embedding.

(2.3.3) Example. Let $D\colon N \times \mathbb{R} \to N$ be a diffeotopy. Then $D^{\#}$ is a diffeomorphism (obviously a bijective immersion). Conversely, if a height preserving diffeomorphism of this type is given and if $D_0 = \mathrm{id}(N)$, then $\mathrm{pr}_1 \circ D^{\#} = D$ is a diffeotopy of N. ◇

(2.3.4) Note. *An isotopy $h\colon M \times \mathbb{R} \to N$ which is constant away from a compact set $K \subset M$ is strict.*

Proof. Since $h^{\#}$ is an injective immersion it suffices to show that $h^{\#}$ is a topological embedding.

Let $U \subset M$ be open and relatively compact and and suppose that h_t is constant in the complement of U. Then $h_0|M \smallsetminus U$ is a homeomorphism onto a closed subset of $h_0(M)$; hence $h_0 \times \mathrm{id} = h^{\#}\colon (M \smallsetminus U) \times [0,1] \to N \times [0,1]$ is a homeomorphism onto a closed subset of $h^{\#}(M \times [0,1])$. By injectivity and compactness also $h^{\#}\colon \overline{U} \times [0,1] \to N \times [0,1]$ is a homeomorphism onto a closed subset. Altogether we see that $h^{\#}$ is a homeomorphism onto a closed subset of the image of $h^{\#}$. □

A vector field X on $M \times \mathbb{R}$ can be decomposed according to the direct decomposition $T_{(x,t)}(M \times \mathbb{R}) = T_x M \times T_t \mathbb{R}$. We then obtain from X two vector fields on $M \times \mathbb{R}$ which we call the M- and the $\mathbb{R}$-component of X. (In a similar manner we can treat arbitrary products $M_1 \times M_2$.) In particular we have on $M \times \mathbb{R}$ the constant vector field with M-component zero and $\mathbb{R}$-component $\frac{\partial}{\partial t}$.

(2.3.5) Proposition. *Let Z be a smooth vector field on $N \times \mathbb{R}$ with $\mathbb{R}$-component $\frac{\partial}{\partial t}$.*

(1) *Suppose Z is globally integrable. Then its flow Φ satisfies*

$$\Phi_t(N \times \{s\}) \subset N \times \{s+t\}, \qquad s,t \in \mathbb{R}$$

and

$$D\colon N \times \mathbb{R} \to N, \quad (x,t) \mapsto \mathrm{pr}_1 \circ \Phi_t(x,0) = D_t(x)$$

is a diffeotopy of N.

(2) *If Z has in addition in the complement of the compact set $C = K \times [c,d]$ the N-component zero, then Z is globally integrable and D is constant in the complement of K.*

Proof. (1) Let $\alpha\colon \mathbb{R} \to N \times \mathbb{R}$ be the integral curve through (y,s). Then $\beta = \mathrm{pr}_2 \circ \alpha$ is the integral curve of $\frac{\partial}{\partial t}$ through s, and therefore the relation $\beta(t) = s + t$ holds. This proves the first inclusion.

We know that D is smooth, and $D(y,0) = \mathrm{pr}_1 \circ \Phi_0(y,0) = \mathrm{pr}_1(y,0) = y$. Moreover, D_t is a diffeomorphism, since a smooth inverse is given by $y \mapsto \mathrm{pr}_1 \circ \Phi_{-t}(y,t)$.

(2) Let $\alpha\colon\ ]a,b[\, \to N \times \mathbb{R}$ be a maximal integral curve. If $b < \infty$, then there exists $t_0 \in\,]a,b[$ such that $\alpha(t) \notin C$ for $t \geq t_0$. As long as the integral curve stays within C it has the form $t \mapsto (x_0, s_0 + t)$. This leads to a contradiction with $b < \infty$. □

(2.3.6) Theorem. *Let $h\colon M \times \mathbb{R} \to N$ be an isotopy which is constant in the complement of a compact set $K \subset M$. Then there exists an isotopy D of N which carries h along and which is constant in the complement of a compact set.*

Proof. We obtain D by the method of 2.3.5. By 2.3.4, h is strict and therefore $P = h^{\#}(M \times \mathbb{R})$ a submanifold. The diffeomorphism $h^{\#}$ transports the vector field $\frac{\partial}{\partial t}$ to a vector field X on P. The $\mathbb{R}$-component of X is $\frac{\partial}{\partial t}$. The set $N_0 = h^{\#}(K \times [0,1])$ is compact in P. Let N_1 be a compact neighbourhood of N_0. We look for an extension Y of P to $N \times \mathbb{R}$ with $\mathbb{R}$-component $\frac{\partial}{\partial t}$ which has away from $N_1 \times [0,1]$ the N-component zero. We can apply 2.3.5 to Y. The resulting diffeotopy carries h along. The vector field Y will be obtained from suitable local data with the help of a partition of unity. In the complement of $N_0 \times [\varepsilon, 1-\varepsilon]$ we take the vector field $\frac{\partial}{\partial t}$. On $N_1^\circ \times\,]0,1[$ we construct an extension of X; this can be done, because the part of P therein is a submanifold. The two part are combined with a partition of unity. □

(2.3.7) Theorem. *Let k_0 and k_1 be collars of M. Let ∂M be compact and K a compact neighbourhood of ∂M in M. Then there exists $\varepsilon > 0$ and a diffeotopy D of M which is constant on $\partial M \cup (M \smallsetminus K)$ and satisfies $D_1 k_0(x,s) = k_1(x,s)$ for $0 \leq s < \varepsilon$.*

Proof. The vector field $\partial/\partial t$ on $\partial M \times [0,1[$ is transported by k_0 and k_1 into vector fields which are defined in a neighbourhood of ∂M in M. Denote them by X_0 and X_1 on the intersection U of these neighbourhoods. On U we have the family of vector fields $X_\lambda = (1-\lambda)X_0 + \lambda X_1$, $\lambda \in [0,1]$. Each member of the family is pointing inwards on ∂M. As in the proof of 2.1.6 we obtain from X_λ a collar k_λ. There exist $\varepsilon > 0$ such that all these collars k_λ, $\lambda \in [0,1]$ are defined on $\partial M \times [0, 2\varepsilon]$, and

$$k\colon (\partial M \times [0,2\varepsilon]) \times [0,1] \to M, \quad (x,s,\lambda) \mapsto k_\lambda(x,s)$$

is a smooth isotopy (of the restrictions) of k_0 to k_1 and have an image in K°.

There exists a diffeotopy of K° which carries along $k|(\partial M \times [0,\varepsilon]) \times [0,1]$ and is constant away from a compact set of K°. We can therefore extend this diffeotopy to M if we use away from K° the constant diffeotopy. The extended diffeotopy has the required property. □

Problems

1. Let M be a connected manifold without boundary of dimension greater that 1. Let $\{y_1,\dots,y_n\}$ and $\{z_1,\dots,z_n\}$ be subsets of M. Then there exist a diffeomorphism $h\colon M \to M$ which is smoothly isotopic to the identity such that $h(y_i) = z_i$ for $1 \le i \le n$. The diffeotopy can be chosen to be constant in the complement of a compact set.

2. For each $t \in \mathbb{R}$ the map

$$h_t\colon\ ]0,\infty[\to \mathbb{R}^2, \quad x \mapsto ((x^{-1}t^2 - x)^2, (x^{-1}t^2 - x)t)$$

is an embedding. (If $t \neq 0$, then $t^{-1}\operatorname{pr}_2 \circ h_t\colon x \mapsto x^{-1}t^2 - x$ is a diffeomorphism $]0,\infty[\to \mathbb{R}$; in the case $t = 0$ the map $\operatorname{pr}_1 \circ h_t\colon x \mapsto x^2$ is an embedding with image $]0,\infty[$.) Hence h_t is a smooth isotopy. But $h^\#$ is not an embedding and therefore h_t not strict. (We have $h^\#(\sqrt{1+t^2} - 1, t) = (4, 2t, t)$, $t \neq 0$. The limit $t \to 0$ is $(4,0,0)$; but $(\sqrt{1+t^2} - 1, t)$ does not converge in $]0,\infty[\times \mathbb{R}$.) The image P of $h^\#$ is not a submanifold, for P is the subset of $Q = \{(x,y,z) \in \mathbb{R}^3 \mid xz^2 = y^2\}$ given by $z \neq 0, x \geq 0$ and $z = 0, x > 0$. The intersection of P with $\{x = a^2 > 0\}$ consists of the two lines $\{(a^2, \pm at, t) \mid t \in \mathbb{R}\}$, and $P \cap \{x > 0\}$ consists of two submanifolds of $\mathbb{R}^3$ with transverse intersection along the positive x-axis. ◇

2.4 Sprays

Let M be smooth manifold with tangent bundle $\pi_M\colon TM \to M$. A vector field $\xi\colon TM \to TTM$ on TM is called ***differential equation of second order*** or ***vector field of second order*** on M if $T\pi_M \circ \xi = \mathrm{id}(TM)$. A vector field ξ

of second order is said to be a ***spray*** on M if for each $s \in \mathbb{R}$ and $v \in TM$ the equality $\xi(sv) = Ts(s\xi(v))$ holds. Here s also denotes the map $s\colon TM \to TM$ which multiplies each tangent vector by the scalar s, and Ts is its differential.

Let $\varphi\colon M \to N$ be a diffeomorphism. We associate to the vector field ξ on TM the transported vector field $\eta = TT\varphi \circ \xi \circ T\varphi^{-1}$ on TN. One verifies from the definitions:

(2.4.1) Note. *η is a vector field of second order (a spray) if and only if this holds for ξ.* □

Let $U \subset M$ be open. By restriction we obtain $\xi_U\colon TU \to TTU$, and ξ_U is a vector field of second order (a spray) if this holds for ξ. Therefore we can study a spray in local coordinates.

Let $U \subset \mathbb{R}^n$ be open. We identify as usual

$$TU = U \times \mathbb{R}^n, \qquad TTU = (U \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n).$$

Then a vector field ξ on TU has the form $(x, v) \mapsto (x, v, f(x, v), g(x, v))$. One verifies:

(2.4.2) Note. *ξ is of second order if and only if $f(x, v) = v$. And a second order ξ is a spray if and only if $g(x, sv) = s^2 g(x, v)$ holds for each $s \in \mathbb{R}$ and each $(x, v) \in U \times \mathbb{R}^n$.* □

The last condition is trivially satisfied if $g = 0$. Hence sprays exist at least locally. Sprays can be globalized with partitions of unity. The geometric meaning of a spray is seen by looking at its integral curves.

(2.4.3) Proposition. *A smooth vector field ξ on TM is a vector field of second order if an only if for each maximal integral curve $\beta_v\colon\ ]a_v, b_v[\ \to TM$ with initial condition $v \in TM$ and projection $\alpha_v = \pi_m \circ \beta_v$ the relation $\alpha_v' = \beta_v$ holds.*

Proof. Let β_v be an integral curve; this means $\xi(v) = T_0\beta_v(\partial/\partial t)$. We apply $T\pi_M$ and obtain with the chain rule

$$\begin{aligned} T_v\pi_M(\xi(v)) &= T_v\pi_M T_0\beta_v(\partial/\partial t) = T_0(\pi_M\beta_v)(\partial/\partial t) \\ &= T_0(\alpha_v)(\partial/\partial t) \overset{(1)}{=} \beta_v(0) = v, \end{aligned}$$

where (1) uses the condition of the proposition.

Conversely, assume $T\pi_M \circ \xi = \mathrm{id}$, and let β_v be an integral curve starting in v. We compute

$$T_s\alpha_v(\partial/\partial t) = T_{\beta_v(s)}\pi T_s\beta_v(\partial/\partial t) = T_{\beta_v(s)}\pi(\xi(\beta_v(s)) = \beta_v(s).$$

We have used: the definition of α; the definition of an integral curve; the assumption about ξ. □

(2.4.4) Proposition. *Let ξ be a smooth vector field of second order on M. Then ξ is a spray if and only if the integral curves have the following properties:*

(1) *For $s,t \in \mathbb{R}$ and $v \in TM$ the following holds: $st \in]a_v, b_v[$ if and only if $t \in]a_{sv}, b_{sv}[$.*

(2) *For $s,t \in \mathbb{R}$ and $v \in TM$ with $st \in]a_v, b_v[$ the equality $\alpha_v(st) = \alpha_{sv}(t)$ holds.*

Proof. Suppose the integral curves have these properties. We differentiate $\alpha_{sv}(t) = \alpha_v(st)$ with respect to t and obtain $\beta_{sv}(t) = s\beta_v(st)$; we differentiate again and obtain, using the chain rule, $\beta'_{sv}(t) = Ts(s\beta'(st))$. For $t = 0$ we obtain $\xi(sv) = Ts(s\xi(v))$.

Conversely, suppose that ξ is a spray. We consider the integral curve $\gamma_v : t \mapsto s\beta_v(st)$ for those t for which $st \in]a_v, b_v[$. Then

$$\gamma'_v(t) = Ts(s\beta'_v(st)) = Ts(s\xi(\beta_v(st))) = \xi(s\beta_v(st)) = \xi(\gamma_v(t)).$$

We have used: chain rule; integral curve; assumption about ξ; definition of γ. The computation shows that γ_v is an integral curve of ξ starting in $\gamma_v(0) = s\beta_v(0) = sv$. Similarly, β_{sv} is an integral curve with the same properties. By uniqueness of integral curves, $\beta_{sv}(t) = s\beta_v(st)$. Hence for each t such that $st \in]a_v b_v[$ the inclusion $t \in]a_{sv}, b_{sv}[$ holds. If $s \neq 0$, we apply the same argument to $1/s$ and obtain the reversed inclusion. For $s = 0$ the situation is clear and causes no problem. We apply the projection to the equality for the β-curves and obtain $\alpha_v(st) = \alpha_{sv}(t)$. □

2.5 The Exponential Map of a Spray

Let $p\colon E \to M$ be a smooth vector bundle. The zero section $i\colon M \to E$ sends $x \in M$ to the zero vector $i(x) \in E_x = p^{-1}(x)$; it is a smooth embedding which we consider as inclusion; we also call the submanifold the $M = i(M)$ as zero section of E. We determine $TE|M$, the restriction of TE to the zero section. We have two types of tangent vectors: The horizontal ones, tangent to M, and the vertical ones, tangent to E_x. This yields a decomposition of $TE|M$ into a Whitney-sum.

The differential $Ti\colon TM \to TE$ has an image in $TE|M$. Since i is an embedding, Ti is a fibrewise injective bundle morphism. Moreover, we have the bundle morphism $j\colon E \to TE|M$; it sends $v \in E_x$ to the velocity vector at $t = 0$ of the curve $t \mapsto tv$ in E_x. (Associated $\mu\colon \mathbb{R} \times E \to E$, $(t,v) \mapsto tv$ belongs $\mu'\colon \mathbb{R} \times E \to TE$.)

(2.5.1) Note. *$\langle j, Ti \rangle\colon E \oplus TM \to TE|M$ is an isomorphism of vector bundles.* □

We apply these considerations to the bundle $\pi_M \colon TM \to M$ and obtain a canonical isomorphism $TM \oplus TM \cong TTM|M$. One has to remember that the two factors TM have a different meaning: The first one comprises the ***vertical*** tangent vectors, the second one the ***horizontal*** tangent vectors.

Now suppose that ξ is a spray on M. The set

$$\mathcal{O}(\xi) = \{v \in TM \mid 1 \in]a_v, b_v[\,\}$$

is an open neighbourhood of the zero section. The ***exponential map*** of the spray is

$$\exp_\xi \colon \mathcal{O}(\xi) \to M, \quad v \mapsto \alpha_v(1).$$

This map is the identity on the zero section. By the preceding remarks

$$T\mathcal{O}(\xi)|M \cong TM \oplus TM.$$

Under this identification the following holds:

(2.5.2) Proposition. *The differential of* $\exp_\xi$ *at the zero section is*

$$T_x \exp_\xi \colon T_x\mathcal{O}(\xi) = T_xM \oplus T_xM \to T_xM, \quad (v, w) \mapsto v + w.$$

Proof. Since $\exp_\xi |M = \mathrm{id}(M)$, the horizontal vectors are mapped identically $(0, w) \mapsto w$. On the vertical summand

$$t \mapsto tv \mapsto \exp_\xi(tv) = \alpha_{tv}(1) = \alpha_v(t)$$

with derivative $\alpha_v'(0) = v$ at $t = 0$; hence the mapping is $(v, 0) \mapsto v$. □

(2.5.3) Theorem. *The differential of* $(\pi, \exp_\xi) \colon \mathcal{O}(\xi) \to M \times M$ *at the zero section is* $T_xM \oplus T_xM \to T_xM \oplus T_xM, (v, w) \mapsto (w, v + w)$.

Proof. Because of 2.5.2, we only have to observe that $T\pi_M$ has the form $(v, w) \mapsto w$, since π_M is constant on fibres and the identity on the horizontal part. □

By 2.5.3 and **??** there exists an open neighbourhood $U(\xi) \subset \mathcal{O}(\xi)$ of the zero section which is mapped under $(\pi, \exp)$ diffeomorphically onto an open neighbourhood $W(\xi) \subset M \times M$ of the diagonal. We can pass to a smaller neighbourhood which has a more appealing form. Choose a Riemannian metric $\langle -, - \rangle$ on TM and a smooth function $\varepsilon \colon M \to]0, \infty[$ such that

$$T^\varepsilon(M) = \{v \in T_xM \mid \|v\| < \varepsilon(x)\}$$

is contained in $U(\xi)$. Then $\pi_M \colon T^\varepsilon(M) \to M$ is a bundle with fibres open disks about the origin.

2.6 Tubular Neighbourhoods

Let $i\colon A \subset M$ be the inclusion of a smooth submanifold. The differential $T_i\colon TA \to TM|A$ is an injective bundle morphism. We think of T_aA as a subspace of T_aM. Fix a Riemannian metric on TM and take the orthogonal complement $N_aA = (T_aA)^\perp$ of T_aA in T_aM. We obtain the sub-bundle NA of $TM|A$, called the ***normal bundle*** of the submanifold. Up to isomorphism it is independent of the Riemannian metric, because it is isomorphic to the quotient bundle of the inclusion $TA \to TM|A$. We have a direct decomposition $TM|A = NA \oplus TA$ which is fibrewise the direct decomposition $T_aM = N_aA \oplus T_aA$ of the subspaces.

Let ξ be a spray on M and $\exp\colon \mathcal{O} \to M$ its exponential map. Then $\mathcal{O}\cap NA$ is an open neighbourhood of the zero section $A \subset \mathcal{O}\cap NA \subset TA$. With respect to the decomposition

$$T_a\mathcal{O} = T_aM \oplus T_aM$$

into the vertical and horizontal part we have

$$T_a|(\mathcal{O} \cap NA) = N_aA \oplus T_aA.$$

Since $T_a \exp$ is on both summands the identity, the differential of the restriction of

$$t = \exp|\mathcal{O} \cap NA\colon \mathcal{O} \cap NA \to M$$

to the summands N_aA and T_aA is the inclusion of these subspaces into T_aM. Therefore we can consider the differential of t at the zero section essentially as the identity. Hence there exists an open neighbourhood U of A in $\mathcal{O} \cap NA$ on which t is an embedding onto an open neighbourhood V of A in M.

(2.6.1) Lemma. *Let U be an open neighbourhood of the zero section in a smooth vector bundle $q\colon E \to A$. Then there exists a fibrewise embedding $\sigma\colon E \to U$ which is the identity on a neighbourhood of the zero section (shrinking of E).*

Proof. We choose a Riemannian metric on E. There exists a smooth function $\varepsilon\colon A \to]0,\infty[$ such that

$$U_{\varepsilon(a)}(a) = \{x \in E_a \mid \|x\| < \varepsilon(a)\} \subset U.$$

Let $\varphi_\eta\colon [0,\infty[\to [0,\eta[$ be a diffeomorphism which is the identity on $[0,\eta/2[$ and which depends smoothly on η. We set $\sigma(v) = \varphi_{\varepsilon(qv)}(\|v\|)\|v\|^{-1}v$. □

A ***tubular map*** for a smooth submanifold $A \subset M$ is a smooth embedding $t\colon NA \to M$ onto an open set $U \subset M$ which is the identity on the zero section. We call the image U of a tubular map a ***tubular neighbourhood*** of A in M. The tubular map transports the bundle projection into a smooth retraction

$r\colon U \to M$. A ***partial tubular map*** maps an open neighbourhood of the zero section of NA diffeomorphically onto an open neighbourhood of A in M (and is the inclusion $A \subset M$ on the zero section). By shrinking, we obtain from a partial tubular neighbourhood a global one. A tubular map is said to be ***strong*** if its differential at the zero section satisfies a further condition to be explained now. The differential of t, restricted to $TNA|A$, is a bundle morphism

$$TNA|A \to TM|A.$$

We compose with the inclusion

$$NA \to NA \oplus TA \cong TNA|A$$

and the projection

$$TM|A \to NA = (TM|A)/TA.$$

If this composition is the identity, we call t strong. Similarly for partial tubular maps. Shrinking 2.6.1 does not affect this property.

(2.6.2) Proposition. *A smooth submanifold $A \subset M$ has a strong tubular map.*

Proof. We have seen in the beginning of this section that the exponential map of a spray yields a strong partial tubular map. □

We transport the bundle projection with a tubular map and obtain:

(2.6.3) Proposition. *A smooth submanifold A of M is a smooth retract of an open neighbourhood.* □

Problems

1. Let $A \subset M$ be a smooth submanifold and $q\colon E \to A$ a smooth vector bundle. Let $\tau\colon E \to M$ be an embedding of E onto an open subset U of M which is the inclusion on the zero section. Then we call τ also a tubular map. This is justified by the fact that E is isomorphic to the normal bundle. The differential of τ yields a smooth bundle isomorphism $T\tau\colon TE|A \to TU|A$ which is the identity on the subbundles TA, and therefore it induces an isomorphism of the quotient bundles $E \to NA$.

2. (Normal bundle of the zero section) Let $q\colon E \to A$ be a smooth vector bundle. The zero section is a smooth submanifold $i\colon A \subset E$. Its normal bundle is E. We had a direct decomposition $TE|A = E \oplus TA$ and therefore a direct complement of TA. A strong tubular map is in this case the identity of E.

3. (Normal bundle of the diagonal) The normal bundle of the diagonal $M = D(M) \subset M \times M$ is isomorphic to the tangent bundle TM. The tangent space $T_{(x,x)}D(M)$ is the diagonal of $T_xM \times T_xM = T_{(x,x)}(M \times M)$, and the bundle $0 \oplus TM$ is a direct complement of the diagonal bundle.

2.7 Morse Functions

Let $U \subset \mathbb{R}^n$ be an open neighbourhood of $0 \in \mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ a smooth function with 0 as a critical point, i.e. the differential $Df(0)$ of f is zero, equivalently, the partial derivatives $D_i f(0)$ of f are zero. The symmetric matrix of the second partial derivatives

$$Hf(0) = D^2 f(0) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0) \right)$$

is called the ***Hesse matrix*** of f. The critical point is ***regular*** or ***non-degenerate*** if this matrix is regular. The differential $Df\colon U \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ has the derivative $D^2 f\colon U \to \mathrm{Hom}(\mathbb{R}^n, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}))$. We identify the latter Hom-space with the vector space of bilinear forms on $\mathbb{R}^n$. The bilinear form $D^2 f(0)$ is described in standard coordinates by the Hesse matrix and called ***Hesse form*** of the critical point. Let $\varphi\colon U \to V$ be a diffeomorphism with $\varphi(0) = 0$. Then 0 is a critical point of $f \circ \varphi$ if and only if it is a critical point of f. The chain rule yields

$$H(f \circ \varphi)(0) = D\varphi(0)^t \cdot Hf(0) \cdot D\varphi(0),$$

provided 0 is a critical point of f. Hence 0 is regular for $f \circ \varphi$ if and only if it is regular for f.

Let M be smooth n-manifold, $f\colon M \to \mathbb{R}$ a smooth function and p a critical point of f. Let $X_p, Y_p \in T_p M$ be given and suppose X, Y are smooth vector fields defined in a neighbourhood of p such that $X(p) = X_p$ and $Y(p) = Y_p$. Then the Lie bracket $X_p(Yf) - Y_p(Xf) = [X,Y]_p f = 0$. The map

$$H_p\colon T_p M \times T_p M \to \mathbb{R}, \quad (X_p, Y_p) \mapsto X_p Y f$$

is therefore independent of the choice of X and Y and a symmetric bilinear form, called the ***Hesse form*** of f in the critical point p. In local coordinates about p the Hesse form of f with respect to the basis $(\frac{\partial}{\partial x_i}|_p)$ is described by the Hesse matrix above.

If H is a symmetric matrix, then the number of negative eigenvalues is the ***index*** of the matrix and the associated bilinear form.

(2.7.1) Proposition. *Let p be a non-degenerate critical point of index i. Then there exists a chart (U, φ, V) centered at p such that $f \circ \varphi^{-1}$ has the form*

$$(y_1, \dots, y_n) \mapsto f(p) - y_1^2 - \cdots - y_i^2 + y_{i+1}^2 + \cdots + y_n^2$$

hat.

Proof. The problem is of a local type, hence we can assume that M is an open ball $U_\varepsilon(0) \subset \mathbb{R}^n$, $p = 0$ and $f(0) = 0$. We can write $f(x) = \sum_{i=1}^n x_i g_i(x)$

with smooth g_i. Since 0 is a critical value, we have $g_i(0) = 0$. We apply a similar procedure to g_i and obtain a presentation $f(x) = \sum_{i,j=1}^n x_i x_j h_{ij}(x)$ with smooth functions h_{ij}. Without essential restriction we can assume that $h_{ij} = h_{ji}$. The matrix $H(x) = (h_{ij}(x))$ at $x = 0$ is the Hesse matrix and regular by assumption. Let $S_n(\mathbb{R})$ the set of symmetric $n \times n$-matrices. The map $\Psi\colon M_n(\mathbb{R}) \to S_n(\mathbb{R}), X \mapsto X^t H X$ is in a neighbourhood of the unit matrix a submersion. Hence there exists an open neighbourhood $U(H)$ of H in $S_n(\mathbb{R})$ and a smooth section $\Theta\colon U(H) \to M_n(\mathbb{R})$ of Ψ. Let U be a neighbourhood of zero such that for $x \in U$ the matrix $H(x)$ is contained in $U(H)$. Then

$$\Theta(H(x))^t \cdot H \cdot \Theta(H(x)) = H(x)$$

holds for $x \in U$ and hence

$$f(x) = \langle\, H(x), x \,\rangle = \langle\, H \cdot \Theta(H(x))x, \Theta(H(x))x \,\rangle.$$

The map $\varphi\colon U \to \mathbb{R}^n, x \mapsto \Theta(H(x))x$ has at 0 the differential $\Theta(H(0)) = E$. Therefore φ is in a neighbourhood of zero a coordinate transformation. In the new coordinates f has the form $f(\varphi^{-1}(u)) = \langle\, Hu, u \,\rangle$. Finally we have to transform H by linear algebra into the correct diagonal form. □

The previous result is called the ***Morse-Lemma***. It implies that a regular critical point is an isolated critical point.

We call f a ***Morse function*** if all its critical points are regular. Let B be a compact manifold with boundary $\partial B = V + W$ the disjoint union of closed manifolds V and W. A smooth function $f\colon B \to [a, b]$ is called a ***Morse function*** of the bordism $(B; V, W)$ if:

(1) $V = f^{-1}(a)$ and $W = f^{-1}(b)$,

(2) The critical points of f are regular and contained in the interior of B.

We show in a moment the existence of Morse functions. For the proof we need a few auxiliary results.

(2.7.2) Lemma. *Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}$ smooth. Then f is a Morse function if and only if $0 \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ is not a critical value of $Df\colon U \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$.*

Proof. The set $K(f)$ of critical points of f is the pre-image of $0 \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ under Df. Let $x \in K(f)$. Then x is a regular critical point if and only if $Df\colon U \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ has at x a bijective differential, i.e., if x is a regular point of Df. If each $x \in K(f)$ is a regular point of Df, then, by definition, $0 \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ is a critical value of Df. □

(2.7.3) Lemma. *Let $f\colon U \to \mathbb{R}$ be smooth and $\lambda \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$. The function $f_\lambda\colon x \mapsto f(x) - \lambda(x)$ is a Morse function if and only if λ is not a critical value of Df. For each λ away from a set of measure zero f_λ is a Morse function.*

Proof. We have $Df_\lambda = Df - \lambda$. Zero is a regular value of Df_λ if and only if λ is a regular value of Df. The second assertion is a consequence of the theorem of Sard. □

In the next lemma we use the sup-norm of the first and second derivatives $\|Df\|_K = \max(|\frac{\partial f}{\partial x_j}(x)| \mid x \in K, 1 \le j \le n)$ and $\|D^2 f\|_K = \max(|\frac{\partial^2 f}{\partial x_i \partial x_j}(x)| \mid x \in K, 1 \le i,j \le n)$.

(2.7.4) Lemma. *Let $K \subset U$ be compact. Suppose $f\colon U \to \mathbb{R}$ has only regular critical points in K.Then there exists a $\delta > 0$ such that smooth functions $g\colon U \to \mathbb{R}$ with*

$$\|Df - Dg\|_K < \delta, \qquad \|D^2 f - D^2 g\|_K < \delta$$

have only regular critical points in K.

Proof. Let $x_1, \dots, x_r$ be the critical points of f in K. There are only a finite number, since K is compact and regular points are isolated. Since in a regular critical point $\det(D^2 f(x_j)) \neq 0$, we can choose $\varepsilon > 0$ such that for each $x \in D_\varepsilon(x_j)$ the determinant $\det(D^2 f(x))$ is non-zero. Then there exists a $\delta > 0$ such that for each g with $\|D^2 f - D^2 g\|_K < \delta$ and each $x \in D_\varepsilon(x_j) \cap K$ also $\det(D^2 g(x)) > 0$; hence the critical points of g in $K \cap D_\varepsilon(x_j)$ are non-degenerate. Since f has no critical points in $L = K \smallsetminus \cup_j U_\varepsilon(x_j)$, there exists $c > 0$ with $\|Df\|_L \ge c$. We require from $\delta > 0$ in addition that for each g with $\|Df - Dg\|_K < \delta$ the norm satisfies $\|Dg\| \ge c/2$; then the critical points of $g|K$ are contained in $\cup_j D_\varepsilon(x_j)$. □

(2.7.5) Theorem. *Every bordism $(B; V, W)$ has a Morse function.*

Proof. There exists a smooth function $g\colon B \to [0,1]$ which has no critical points in a neighbourhood of ∂B and satisfies $V = g^{-1}(0)$ and $W = g^{-1}(1)$. A function of this type certainly exists on disjoint collar neighbourhoods of V and W; by a partition of unity argument this (??) function is then extended to B (compare **??**).

Let U be an open neighbourhood of ∂B on which g has no critical points. Let P be an open neighbourhood of ∂B with closure contained in U. Let $(U_1, \dots, U_r)$ be an open covering of $B \smallsetminus P$ by chart domains $U_j \subset B \smallsetminus \partial B$ and $(C_1, \dots, C_r)$ a covering of $B \smallsetminus P$ by compact sets C_i with $C_i \subset U_i$.

Suppose that $g\colon B \to [0,1]$ with $g^{-1}(0) = V$ and $g^{-1}(1) = W$ has on $\overline{P} \cup C_1 \cup \ldots \cup C_i$ for $0 \le i < r$ only regular critical points. We have already seen that such a function exists in the case that $i = 0$. Now choose compact sets Q and R such that

$$C_{i+1} \subset Q^\circ \subset Q \subset R^\circ \subset R \subset U_{i+1}$$

and then choose a smooth function $\lambda\colon B \to [0,1]$ such that $\lambda(Q) \subset \{1\}$ and $\lambda(B \smallsetminus R^\circ) \subset \{0\}$. Let $h_{i+1}\colon U_{i+1} \to V_{i+1}$ be a chart and $L\colon \mathbb{R}^n \to \mathbb{R}$ a linear map. Now consider

$$h\colon B \to \mathbb{R}, \quad b \mapsto g(b) + \lambda(b)L(h_{i+1}(b)).$$

The maps h and g only differ on R. In order to study the critical points it suffices to investigate its composition with h_{i+1}^{-1}, and then we see by using **??**, that h has for sufficiently small L on $(\overline{P} \cup C_1 \cup \ldots \cup C_i) \cap R$ only regular critical points. Since R has a finite distance from ∂B, we still have for sufficiently small L that $h^{-1}(0) = V$, $h^{-1}(1) = W$, and $h(B) \subset [0,1]$. Altogether we obtain an h that has on $\overline{P} \cup C_1 \cup \ldots \cup C_i$ only regular critical points. By **??** we can choose L such that h has on C_{i+1} only regular critical points. This finishes the induction step. □

(2.7.6) Theorem. *Let f be a Morse function for the triple $(B; V, W)$ with critical points $p_1, \ldots, p_k$. Then there exists a Morse function g with the same critical points such that $g(p_i) \neq g(p_j)$ for $i \neq j$.*

Proof. We begin by changing f into f_1 such that for $i \neq 1$ the inequalities $f_1(p_1) \neq f_1(p_i)$ hold. For this purpose let N be a compact neighbourhood of p_1 which is contained in $B \smallsetminus \partial B$ and which does not contain the p_i for $i \neq 1$. Let $\lambda\colon B \to [0,1]$ be a smooth function which is 0 away from N and 1 in a neighbourhood of p_1. Choose $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$ the function $f + \varepsilon\lambda$ assumes values in $[0,1]$ and satisfies $f_1(p_1) \neq f_1(p_i)$ for $i \neq 1$.

With a fixed Riemannian metric on B we form the gradient fields of f and f_1. The zeros of the gradient fields are the critical points. Let K be the closure of $\lambda^{-1}\,]0,1[$. Away from K we have $\operatorname{grad} f_1 = \operatorname{grad} f$ and hence the critical points are the same. They are thus also regular for f_1. Since K does not contain a critical point of f, there exists $c > 0$ such that $c \leq \|\operatorname{grad} f(x)\|$ for each $x \in K$. Choose $d > 0$ such that $\|\operatorname{grad} \lambda(x)\| \leq d$ in K. Let $0 < \varepsilon < \min(\varepsilon_1, c/d)$. Then on K

$$\|\operatorname{grad}(f + \varepsilon\lambda)\| \geq \|\operatorname{grad} f\| - \varepsilon\|\operatorname{grad} \lambda\| \geq c - \varepsilon d > 0.$$

Therefore f_1 has on K no further critical points. The other points p_i are treated in a similar manner (induction). □

The theorems **??** and **??** have the following consequence. Let B be a bordism between V and W. We choose a Morse function according to **??** and choose an indexing of the critical points such that $g(p_i) < g(p_{i+1})$. Let $g(p_i) < c_i < g(p_{i+1})$. Then

$$g^{-1}[0, c_1], g^{-1}[c_1, c_2], \ldots, g^{-1}[c_{r-1}, 1]$$

are bordisms which have a Morse function with a single critical point. Thus each bordism is a "composition" of such elementary bordisms.

We give another proof for the existence of Morse function for submanifolds of Euclidean spaces without taking care of boundaries.

(2.7.7) Proposition. *Let $M \subset \mathbb{R}^N$ be a smooth n-dimensional submanifold and $f\colon M \to \mathbb{R}$ a smooth function. Then for almost all $\lambda \in \mathrm{Hom}(\mathbb{R}^N, \mathbb{R})$ the function $f_\lambda\colon M \to \mathbb{R}, x \mapsto f(x) - \lambda(x)$ is a Morse function.*

Proof. If $(U_\nu \mid \nu \in \mathbb{N})$ is an open cover of M, then it suffices to verify the assertion for each restriction $f|U_n u$, since a countable union of sets of measure zero has measure zero.

We therefore assume that M is the image of a smooth embedding $\varphi\colon U \to \mathbb{R}^N$ of an open subset $U \subset \mathbb{R}^n$. Then f_λ is a Morse function if and only if $f_\lambda \circ \varphi$ is a Morse function. Set $g = f \circ \varphi$. Then $f_\lambda \circ \varphi$ has the form

$$x \mapsto g(x) - \textstyle\sum_{j=1}^N \lambda_j \varphi_j(x),$$

if $\varphi = (\varphi_1, \dots, \varphi_N)$ and $\lambda(x_1, \dots, x_n) = \sum_j \lambda_j x_j$.

The critical point of this map are the x for which

$$Dg(x) - \textstyle\sum_j \lambda_j D\varphi_j(x) = 0.$$

We asks for the $(\lambda_1, \dots, \lambda_n)$ such that zero is a regular value of

$$U \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}), \quad x \mapsto Dg(x) - \textstyle\sum_j \lambda_j D\varphi_j(x).$$

So let us consider

$$F\colon U \times \mathbb{R}^N \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}), \quad (x, (\lambda_j)) \mapsto Dg(x) - \textstyle\sum_j \lambda_j D\varphi_j(x).$$

We keep x fixed. The linear map $(\lambda_j) \mapsto Dg(x) - \sum_j \lambda_j D\varphi_j(x)$ is surjective, since $D\varphi$ has rank n. Hence this map is a submersion and therefore also F. By **??** we see that for all most all λ zero is a regular value of F_λ. □

(2.7.8) Example. Let $a_0 < a_1 < \ldots < a_n$. Then

$$f\colon S^n \to \mathbb{R}, \quad (x_0, \dots, x_n) \mapsto \textstyle\sum_i a_i x_i^2$$

is a Morse function. The critical points are the unit vectors $(\pm e_j \mid 0 \le j \le n)$, and the index of $\pm e_j$ is j. This function induces a Morse function $g\colon \mathbb{R}P^n \to \mathbb{R}$ with $n+1$ critical points of index $0, 1, \dots, n$. ◇

2.8 Elementary Bordisms

We show in this section that an elementary bordism is diffeomorphic to a standard model. We begin by describing the standard model.

Let V be a (compact) manifold (without boundary) of dimension $n-1$. Suppose $a+b=n, a\geq 1, b\geq 1$. Let $\varphi\colon S^{a-1}\times E^b\to V$ be a smooth embedding. In the disjoint sum $(V\smallsetminus\varphi(S^{a-1}\times 0))+E^b\times S^{b-1}$ we identify

$$\varphi(u,\Theta v)\sim(\Theta u,v),\qquad u\in S^{a-1},\ v\in S^{b-1},\ 0<\Theta<1.$$

The quotient has a canonical structure of a smooth manifold (see **??**). We denote the result by $\chi(V,\varphi)$ and say: $\chi(V,\varphi)$ (or a diffeomorphic manifold V') is obtained from V by ***elementary surgery of type*** (a,b). In general the result depends on the choice of φ.

(2.8.1) Remark. One can get back V by an inverse process by an elementary surgery of type (b,a) applied to $\chi(V,\varphi)$. Namely by construction we have an embedding $\tilde\varphi\colon E^a\times S^{b-1}\to\chi(V,\varphi)$ ◇

(2.8.2) Proposition. *There exists a bordism $(H(V,\varphi);V,\chi(V,\varphi))$ with a Morse function which has a single critical point of index a.*

Proof. We first describe a standard object which is used in the construction of $H(V,\varphi)$. The map

$$\alpha\colon\,]0,\infty[\,\times\,]0,\infty[\,\to\,]0,\infty[\,\times\mathbb{R},\quad (x,y)\mapsto(xy,x^2-y^2)$$

is a diffeomorphism. We denote by $(c,t)\mapsto(\gamma(c,t),\delta(c,t))$ the inverse diffeomorphism. We claim that

$$\begin{aligned}\Psi\quad &:\quad (\mathbb{R}^a\smallsetminus 0)\times(\mathbb{R}^a\smallsetminus 0)\to S^{a-1}\times(\mathbb{R}^b\smallsetminus 0)\times\mathbb{R},\\ \Psi(x,y)\quad &=\quad \left(\frac{x}{\|x\|},\frac{y}{2\|y\|}\operatorname{arsinh}(2\|x\|\|y\|),-\|x\|^2+\|y\|^2\right)\end{aligned}$$

is a diffeomorphism. An inverse is defined by

$$(u,v,t)\mapsto\left(u\gamma(c,t),\frac{v}{\|v\|}\delta(c,t)\right)$$

with $c=\frac12\sinh 2\|u\|\|v\|$. For $d>0$ we set

$$L^{a,b}(d)=\{(x,y)\in\mathbb{R}^a\times\mathbb{R}^b\mid -1\leq-\|x\|^2+\|y\|^2\leq 1,\ 2\|x\|\|y\|<\sinh 2d\}$$

and $L^{a,b}(1)=L^{a,b}$ One verifies that Ψ induces by restriction a diffeomorphism

$$\Psi\colon L^{a,b}(d)\cap((\mathbb{R}^a\smallsetminus 0)\times(\mathbb{R}^b\smallsetminus 0))\to S^{a-1}\times(E^b(d)\smallsetminus 0)\times D^1.$$

In the disjoint sum $L^{a,b}+(V\smallsetminus\varphi(S^{a-1}\times 0))\times D^1$ we use the identification

$$\begin{array}{ccc}(V\smallsetminus\varphi(S^{a-1}\times 0))\times D^1 & & L^{a,b}\\ \uparrow{\scriptstyle\varphi\times\mathrm{id}} & & \uparrow{\scriptstyle\cup}\\ S^{a-1}\times(E^b\smallsetminus 0)\times D^1 & \xleftarrow{\ \Psi\ } & L^{a,b}\cap((\mathbb{R}^a\smallsetminus 0)\times(\mathbb{R}^b\smallsetminus 0)).\end{array}$$

The result is a bordism $H(V,\varphi)$. This manifold has two boundary pieces V and $\chi(V,\varphi)$. The first piece is given by $V \to H(V,\varphi)$

$$z \mapsto \begin{cases} (z,-1) & z \in V \smallsetminus \varphi(S^{a-1}\times 0) \\ (u\cosh\Theta, v\sinh\Theta) \in L^{a,b} & z = \varphi(u,\Theta v), \|u\| = \|v\| = 1. \end{cases}$$

Observe that $\Psi(u\cosh\Theta, v\sinh\Theta) = (u,\Theta v,-1)$. The second piece is given by $\chi(V,\varphi) \to H(V,\varphi)$

$$\begin{array}{rcl} V \smallsetminus \varphi(S^{a-1}\times 0) \ni z & \mapsto & (z,1) \\ E^a \times S^{b-1} \ni (\Theta u, v) & \mapsto & (u\sinh\Theta, v\cosh\Theta). \end{array}$$

A Morse function $f\colon H(V;\varphi) \to [-1,1]$ with the desired properties is defined by

$$\begin{array}{rcll} f(x,c) & = & c & (x,c) \in (V \smallsetminus \varphi(S^{a-1}\times 0)) \times D^1 \\ f(x,y) & = & -\|x\|^2 + \|y\|^2 & (x,y) \in L_{a,b}. \end{array}$$

□

(2.8.3) Proposition. *Let $f\colon B \to [a,b]$ be a Morse function with a single critical point. Then B is diffeomorphic to an elementary bordism $H(V,\varphi)$.*

Proof. Without essential restriction we can assume that $[a,b] = [-d,d]$ and that the critical point p has the value $f(p) = 0$. By the Morse lemma **??** we choose a local parametrization $\alpha\colon W \to U$, centered at p, such that $f\alpha(x,y) = f^{a,b}(x,y) = -\|x\|^2 + \|y\|^2$ for $(x,y) \in W \subset \mathbb{R}^{a,b}$. Let $s_\lambda\colon \mathbb{R}^n \to \mathbb{R}^n$ be the scalar multiplication by λ. For sufficiently small $\lambda > 0$ the inclusion $s_\lambda L^{a,b} \subset W$ holds. The function $\lambda^{-2}f\colon B \to [-\lambda^{-2}d, \lambda^2 d]$ has the critical point p and with respect to the chart $\alpha \circ s_\lambda$ the form $f^{a,b}$. Hence we can assume without essential restriction that $L^{a,b} \subset W$. Then $d \geq 1$ and $f^{-1}[0,1]$ is diffeomorphic to B. Therefore we assume $d = 1$ and set $B_\pm = f^{-1}(\pm 1)$ for the boundary pieces of B.

Let $M^{a,b} = \alpha(L^{a,b}) \subset B$. This is an open submanifold with boundary of B. We had constructed a diffeomorphism Ψ in **??**. The boundary of $L^{a,b}$ consists of the pieces $L^{a,b}_\pm = \{(x,y) \in L^{a,b} \mid f^{a,b}(x,y) = \pm 1\}$, and $\alpha(L^{a,b}_\pm) = M^{a,b}_\pm \subset B_\pm$. The coordinates

$$L^{a,b}_0 = L^{a,b} \cap (\mathbb{R}^a \times 0 \cup 0 \times \mathbb{R}^b)$$

are contained in the source of Ψ. The intersection $L^{a,b}_{0,-} = L^{a,b}_0 \cap L^{a,b}_-$ is the sphere $S^{a-1}\times 0$; and $L^{a,b}_{0,+}$ is the sphere $0 \times S^{b-1}$. The images under α are denoted $S_\pm \subset B_\pm$. We have a diffeomorphism

$$S^{a-1} \times E^b \to L^{a,b}_-, \quad (u,\Theta v) \mapsto (u\cosh\Theta, v\sinh\Theta)$$

which extends the inverse $\Psi^{-1}|S^{a-1}\times(E^b \smallsetminus 0)\times\{-1\}$ to $S^{a-1}\times E^b \times \{-1\}$. We compose it with α and obtain an embedding $\varphi\colon S^{a-1}\times E^b \to B_-$, and $\varphi(S^{a-1}\times\{0\}) = S_-$.

We use this embedding to form $H(B_-,\varphi)$ and claim that the result is diffeomorphic to B. We recall that $H(B_-,\varphi)$ is defined as a quotient of $(B_- \smallsetminus S_-) \times D^1 + L^{a,b}$. We construct the diffeomorphism $D\colon H(B_-,\varphi) \to B$ separately on the two summands. On $L^{a,b}$ we let D be the local parametrization α. On $(B_- \smallsetminus S_-) \times D^1$ we will obtain D by integrating a suitable vector field X on $B \smallsetminus M_0^{a,b}$ (note $M_0^{a,b} = \alpha(L_0^{a,b})$).

We choose on $M^{a,b}(2) = \alpha(L^{a,b}(2))$ the vector field X_0 which corresponds under $\Psi\circ\alpha^{-1}$ to the constant vector field $\partial/\partial t$ of the factor D^1. On $B\smallsetminus M_0^{a,b}$ we have the normed gradient field X_1 of f, i.e., a vector field which satisfies $X_1 f = 1$. We choose a smooth partition of unity τ_0, τ_1 subordinate to the open covering $U_0 = M^{a,b}(2)$ and $U_1 = B \smallsetminus \overline{M}^{a,b}$ and use it to define $X = \tau_0 X_0 + \tau_1 X_1$. This vector field agrees on $M^{a,b}\smallsetminus M_0^{a,b}$ with X_0. For $(x,t) \in (B_-\smallsetminus S_-)\times D^1$ we now set $D(x,t) = k_u(t+1)$ where k_u is the integral curve of X with initial condition $x = k_u(0)$. We have to verify that these conditions yield a diffeomorphism of $(B_- \smallsetminus S_-) \times D^1$ with $B \smallsetminus M_0^{a,b}$.

The curve k_u has interval of definition $[0,2]$ and ends in $B_+ \smallsetminus S_+$. this holds by construction, if the curve begins in $M_-^{a,b}$. The curve stays inside $M^{a,b}$, and this set is covered by such integral curves. The remaining happens in the compact complement of $M^{a,b}$, and we can argue as in **??**. □

We can express **??** with more precision: There exist a diffeomorphisms $B \to H(V,\varphi)$ and $[a,b] \to [-1,1]$ which transform the given Morse function on B into the Morse function of the standard model **??**.

2.9 The Mapping Degree

Let M and N be closed, oriented, smooth n-manifolds ($n \geq 1$). We assume that N is connected. We associate to each map $f\colon M \to N$ an integer $d(f)$, called the ***degree*** of f, which only depends on the homotopy class of f.

Let $y \in N$ be a regular value of a smooth map $f\colon M \to N$. For $f(x) = y$ we set $d(f,x,y) = 1$ if the differential $T_x f$ respects the orientations; otherwise $d(f,x,y) = -1$. We set

(2.9.1) $$d(f,y) = \textstyle\sum_{x\in f^{-1}(y)} d(f,x,y).$$

We will show that this integer does not depend on the choice of y and is an invariant of the homotopy class of f.

Let y be a regular value of f. Then, by the inversion theorem of calculus, there exist open connected neighbourhoods V of y and open neighbourhoods $U(x)$ of x such that

$$f^{-1}(V) = \textstyle\coprod_{x\in f^{-1}(y)} U(x)$$

and $f\colon U(x) \to V$ is a diffeomorphism. This implies:

(2.9.2) Note. *There exists an open neighbourhood V of y such that each $z \in V$ is a regular value and satisfies $d(f,y) = d(f,z)$.* □

(2.9.3) Theorem. *Let B be a compact smooth oriented manifold B with boundary M. Let $F\colon B \to N$ be smooth and y a regular value of F and $F|\partial B = f$. Then $d(f,y) = 0$.*

Proof. Let $J = F^{-1}(y)$. Then $J \subset B$ is a compact submanifold of type I of B with $\partial J = J \cap M = f^{-1}(y)$. Let J carry the pre-image orientation and ∂J the induced boundary orientation. The latter is a function $\epsilon\colon \partial J \to \{\pm 1\}$. A component of the compact 1-manifold J with non-empty boundary is diffeomorphic to $[0,1]$, see (??). From this fact we see $\sum_{x \in \partial J} \epsilon(x) = 0$. In our case we verify from the definitions $\epsilon(x) = (-1)^n d(f,x,y)$ for each $x \in \partial J$. These facts imply the assertion. □

(2.9.4) Corollary. *Let $\partial B = M_0 - M_1$, $F|M_i = f_i$, and assume the hypothesis of* 2.9.3*. Then $d(f_0,y) = d(f_1,y)$.*

(2.9.5) Proposition. *Let $f_0, f_1\colon M \to N$ be smoothly homotopic. Let y be a regular value of f_0 and f_1. Then $d(f_0,y) = d(f_1,y)$.*

Proof. Let $H\colon M \times I \to N$ be a smooth homotopy between f_0 and f_1. By the theorem of Sard, in each neighbourhood of y there exist regular values z of H which are also regular values of $H|M \times \partial I$. The assertion, for z instead of y, follows now from **??**, and **??** then implies the assertion for y. □

In the further development of the mapping degree we use the following two technical results 2.9.6 and 2.9.7. They will be proved later.

(2.9.6) Proposition. *Each continuous map $f\colon M \to N$ is homotopic to a smooth map. If f_0 and f_1 are smooth maps and homotopic, then there exists a smooth homotopy between them.* □

(2.9.7) Proposition. *Let N be a connected smooth k-manifold of dimension $k > 1$. Let $\{y_1, \dots, y_n\}$ and $\{z_1, \dots, z_n\}$ be subsets of N with n elements. Then there exists a smooth homotopy $H\colon N \times [0,1] \to N$ such that:*

(1) $H_0 = \mathrm{id}$.
(2) *H_t is for each $t \in [0,1]$ a diffeomorphism.*
(3) *$H_1(y_j) = z_j$ for $1 \le j \le n$.*
(4) *H_t is constant in the complement of a compact set.*

In case $n = 1$, we can also allow $k = 1$. If N is oriented, then H_t is orientation preserving. □

A homotopy H with the properties (1) and (2) of the previous proposition will be called a ***smooth isotopy*** of the identity.

(2.9.8) Theorem. *Let y and z be regular values of f. Then $d(f, y) = d(f, z)$.*

Proof. There exists a smooth isotopy $H \colon N \times I \to N$ of the identity to a map h with $h(z) = y$. Then y is a regular value of hf. We have $d(f, z) = d(hf, y)$, for we have $f^{-1}(z) = (hf)^{-1}(y)$, and the corresponding sums 2.9.1 are equal, since $T_z h$ respects the orientation. □

We now define the degree map

$$d \colon [M, N] \to \mathbb{Z}$$

as follows. Let $f \colon M \to N$ be given. Choose a smooth map g which is homotopic to f. By **??**, the integer $d(g, y)$ does not depend on the the choice of y and we denote it by $g(y)$. By **??** and **??**, a different choice of g leads to the same integer. Therefore we define the ***degree*** $d(f)$ of f as $d(g)$ for a smooth map g homotopic to f. By construction, $d(f)$ only depends on the homotopy class of f. This definition of the degree gives also a means of computation; we just have to find a situation where **??** applies. The next theorem is immediate from the definitions.

(2.9.9) Theorem. *The degree has the following properties:*

(1) *If we change the orientation in one of the manifolds, then the degree changes its sign.*
(2) *$d(fg) = d(f)d(g)$, whenever the three degrees are defined.*
(3) *A diffeomorphism has degree 1 or -1.*
(4) *Let $t \colon M \times N \to N \times M$, $(x, y) \mapsto (y, x)$. Let the products carries the product orientation. Then t has the degree $(-1)^{nm}$, $m = \dim M$, $n = \dim N$.*
(5) **??** *and* **??** *hold for the degree.*
(6) *If M is the disjoint union of M_1 and M_2, then $d(f) = d(f|M_1) + d(f|M_2)$.*

We use the notion of degree to define the winding number. Let M be a closed, connected, oriented n-manifold. Let $f \colon M \to \mathbb{R}^{n+1}$ be given and assume that a is not contained in the image of f. The ***winding number*** $W(f, a)$ of f with respect to a is the degree of the map

$$p_{f,a} = p_a \colon M \to S^n, \quad x \mapsto N(f(x) - a)$$

with the norm map $N \colon \mathbb{R}^{n+1} \smallsetminus 0 \to S^n$, $x \mapsto \|x\|^{-1}x$. If f_t is a homotopy such that $a \notin \mathrm{image} f_t$ for all t, then $W(f_t, a) = W(f_0, a)$.

(2.9.10) Example. The map $f \colon S^n \to \mathbb{R}^{n+1} \smallsetminus 0$, $x \mapsto Ax$ has for each matrix $A \in GL_{n+1}(\mathbb{R})$ the winding number $\det A$. In each path-component of the group of $GL_{n+1}(\mathbb{R})$ lies an orthogonal matrix. Hence we can assume that $p_{f,0}$ is homotopic to $S^n \to S^n$, $x \mapsto Bx$ for an orthogonal B. Now use the definition **??**. ◇

(2.9.11) Theorem. *Let M be the oriented boundary of the compact, oriented manifold B. Suppose $F\colon B \to \mathbb{R}^{n+1}$ is smooth and has 0 as a regular value. Then*

$$W(f,0) = \textstyle\sum_{x\in P} \epsilon(F,x), \quad P = F^{-1}(0), \quad f = F|M.$$

Here $\epsilon(F,x) \in \{\pm 1\}$ is the orientation behaviour of the differential

$$T_xF\colon T_xB \to T_0\mathbb{R}^{n+1},$$

i.e., $\epsilon(F,x) = 1$ if and only if the differential preserves the orientation.

Proof. For each $x \in P$ we choose $D(x) \subset B \smallsetminus \partial B$, diffeomorphic to a disk D^{n+1} in local coordinates centered at x. We assume that the $D(x)$ are pairwise disjoint. Set $G(x) = NF(x)$; this is defined on the complement $C = B \smallsetminus \bigcup_{x\in P} D(x)$. By (??), we have

$$d(G|\partial B) = \textstyle\sum_{x\in P} d(G|\partial D(x)).$$

Thus it suffices to show

$$d(G|\partial D(x)) = \epsilon(F,x).$$

We use a suitable positive chart about x and reduce the problem to the situation that $D(x) = D^{n+1} \subset \mathbb{R}^{n+1}$ and $F\colon D^{n+1} \to \mathbb{R}^{n+1}$ is smooth with regular value 0 and $F^{-1}(0) = \{0\}$. We define a smooth homotopy

$$H\colon S^n \times I \to \mathbb{R}^{n+1} \smallsetminus 0$$

by

$$H(x,t) = \begin{cases} t^{-1}F(tx) & \text{for} \quad t > 0 \\ DF(0)(x) & \text{for} \quad t = 0 \end{cases}$$

Hence $W(F|S^n, 0) = W(DF(0)|S^n, 0)$. By **??**, this integer equals $\epsilon(F,0) = \det DF(0)$. □

2.10 The Theorem of Hopf

In this section we prove the theorem of Hopf **??** which states that the degree of a map into the n-sphere characterizes the homotopy class of this map.

(2.10.1) Theorem. *Let M be the oriented boundary of the compact, connected, oriented manifold B. Let $f\colon M \to S^n$ be a map of degree zero. Then f admits an extension to B.*

Proof. The inclusion $M = \partial B \subset B$ is a cofibration. If we can extend f, then we can extend any homotopic map. Thus we will assume that f is smooth.

In section one we have seen that we can extent $f\colon M \to S^n \subset \mathbb{R}^{n+1}$ to a smooth map $\phi\colon B \to \mathbb{R}^{n+1}$. By the isotopy theorem (??), we can assume that 0 is a regular value of ϕ and that $\phi^{-1}(0)$ is contained in an open set $U \subset B \smallsetminus \partial B$ which is diffeomorphic to $\mathbb{R}^{n+1}$. Let $B_r \subset U$ be the part which is diffeomorphic to the ball $D_r \subset \mathbb{R}^{n+1}$ of radius r about the origin, and choose r such that $\phi^{-1}(0) \subset B_r \smallsetminus \partial B_r$. By **??**, f and $\varphi|\partial B_r$ have the same winding number. Since the winding number of maps $S^n \to \mathbb{R}^{n+1} \smallsetminus 0$ characterizes the homotopy class of such maps, $\varphi|\partial B_r$ is null-homotopic and has therefore an extension to B_r. We use this extension and combine it with $\varphi|B \smallsetminus \overset{\circ}{B}_r$ to obtain an extension of f into $\mathbb{R}^{n-1} \smallsetminus 0$. We now combine with the norm retraction N and obtain the desired extension of f. □

(2.10.2) Theorem. *Let M be a closed, oriented, connected, smooth manifold. Then the degree map $d\colon [M, S^n] \to \mathbb{Z}$ is a bijection.*

Proof. Suppose $f_0, f_1\colon M \to S^n$ have the same degree. Together they yield a map $M + (-M) \to S^n$ of degree zero. By the previous theorem, we can extend this map to $M \times I$. This argument shows the injectivity of the degree map. For the surjectivity we have to construct maps of a given degree. We construct a map of degree 1 as follows. Choose a smooth embedding $\psi\colon \mathbb{R}^n \to M$. Map the subset $\psi(D^n)$ by ψ^{-1} and a homeomorphism $\varphi\colon D^n/S^{n-1} \cong S^n$ to S^n which is smooth in the interior, and map the complement of $\psi(D^n)$ to the point $\varphi(\{S^{n-1}\})$. The point corresponding to $0 \in D^n/S^{n-1}$ has a single pre-image and is a regular value. Thus this map has degree ± 1. If necessary, we compose with a self-map of S^n and realize the degree 1. The same process can be applied to several disjoint cells, and thus each integer can be realized as a degree. Or we compose with a self-map of S^n with given degree and use **??**. □

2.11 One-dimensional Manifolds

(2.11.1) Theorem. *A connected smooth one-dimensional manifold M without boundary is diffeomorphic to $\mathbb{R}$ or S^1.*

We formulate some steps of the proof as a lemma. Since M has a countable basis, there exists an atlas $\{(U_j, h_j, V_j) \mid j \in J\}$ such that: J is countable; V_j is an open interval; for $i \neq j$ the intersection $U_i \cap U_j$ is different from U_i and U_j. A suitable normalization allows us to choose V_j as a given interval. If $U_i \cap U_j \neq \emptyset$, then $h_i(U_i \cap U_j) = V_i^j$ is a non-empty open subset of the interval V_i

and therefore a disjoint union of open subintervals, called components. Suppose $a < b < c$; then we call b an interior and c an exterior end of the subinterval $]b,c[$ of $]a,c[$.

(2.11.2) Lemma. *No component of V_i^j has both of its end points contained in V_i. Hence V_i^j has at most two components, and each component has one end point contained in V_i.*

Proof. $U_i \cup U_j$ is homeomorphic to the space Z which is obtained from $V_i + V_j$ by the identification $g_i^j = h_j h_i^{-1}\colon V_i^j \to V_j^i$. The space Z is a Hausdorff space. If a component of V_i^j would have both end points in V_i, then the image of $V_i^j \to V_i \times V_j$, $x \mapsto (x, g_i^j(x))$ would not be closed, and this contradicts (??). □

From the Hausdorff property one also obtains:

(2.11.3) Lemma. *The map g_i^j sends each component of V_i^j diffeomorphic onto a component of V_j^i such that an interior end point corresponds to an exterior end point, and conversely.* □

(2.11.4) Lemma. *If V_i^j and hence V_j^i has two components, then $M = U_i \cup U_j$, and M is compact.*

Proof. If $U_i \cup U_j$ is compact, then this subset is open and closed and therefore equals M, since M is connected. Let $K_i \subset U_i$ be compact subsets such that $h_i(K_i)$ is a closed interval which intersects both components of V_i^j. Then $M = K_1 \cup K_2$, and hence M compact. □

(2.11.5) Lemma. *Let $\alpha\colon W \to W$ be an increasing diffeomorphism of $W =]0,1[$. Let $0 < \varepsilon < 1$ be fixed. Then there exists $\eta \in]\varepsilon, 1[$ and a diffeomorphism $\delta\colon W \to W$ such that $\delta(x) = \alpha(x)$ for $x \le \varepsilon$ and $\delta(x) = x$ for $x \ge \eta$.*

Proof. Let $\lambda\colon \mathbb{R} \to \mathbb{R}$ be the C^∞-function $\lambda(x) = 0$ for $x \le 0$ and $\lambda(x) = \exp(-x^{-1})$ for $x > 0$. For $a < b$ we set

$$\psi_{a,b}(x) = \frac{\lambda(x-a)}{\lambda(x-a) + \lambda(b-x)}.$$

Choose $0 < M < 1$ and $\varepsilon < \zeta < 1$ such that for $x < \zeta$ the inequality $\alpha(x) < M$ holds. Write $\eta = \psi_{\varepsilon,\zeta}$. Then $\beta(x) = \alpha(x)(1-\eta(x)) + M(\eta(x))$ is a C^∞-function which coincides for $x \le \varepsilon$ with α and has constant value M for $x \ge \zeta$, and for $x < \zeta$ it is strictly increasing. Let $\max(\zeta, M) < \eta < 1$. In a similar manner one constructs a function γ which is zero for $x \le \varepsilon$, equals $x - M$ for $x \ge \eta$, and is strictly increasing for $x \ge \varepsilon$. The function $\delta = \beta + \gamma$ has the desired properties. □

(2.11.6) Lemma. *Suppose V_i^j is connected. After suitable normalization we can assume that $V_i =]0,2[$, $V_j =]1,3[$ and $V_i^j =]1,2[$. Then there exists a diffeomorphism $\Psi\colon U_i \cup U_j \to]0,3[$ which coincides on U_i with h_i.*

Proof. $U_i \cup U_j$ is via h_i, h_j diffeomorphic to $A =]0,2[\cup_\varphi]1,3[$ with $\varphi = g_i^j$. Therefore is suffices to find a diffeomorphism $A \to]0,3[$ which is on $]0,2[$ the identity. Suppose we have some diffeomorphism $\alpha\colon A \to]0,3[$ which maps $]0,2[$ onto itself; then we compose it with a diffeomorphism which coincides on $]0,2[$ with α^{-1}. Thus it suffices to find some α. For this purpose we choose an increasing diffeomorphism $\Phi\colon]1,2[\to]1,2[$ which coincides on $]1,1+\varepsilon[$ with φ^{-1} and is the identity on $]2-\eta,2[$. Let $\Phi_2\colon]1,3[\to]1,3[$ be the extension of Φ by the identity and $\Phi_1\colon]0,2[\to]0,2[$ the extension of $\Phi \circ \varphi$ by the identity. Then $\langle \Phi_1, \Phi_2 \rangle$ factorizes over a diffeomorphism $\alpha\colon]0,2[\cup_\varphi]1,3[\cong]0,3[$. □

Proof. Suppose M is not compact. Let $U_1, \ldots, U_k$ be chart domains for which we have a diffeomorphism $\varphi_k\colon W_k = U_1 \cup \ldots \cup U_k \to]a, a+k+1[$. If these chart domains don't exhaust M, then there exists a further chart domain, say U_{k+1} such that $C = U_k \cap U_{k+1} \neq \emptyset$ (with suitable indexing). In the case that C is mapped under φ_k onto the upper end of $]a, a+k+1[$, we can by the method of the previous lemma extend φ_k to a diffeomorphism φ_{k+1} of $W_k \cup U_{k+1}$ with $]a, a+k+2[$. This settles, inductively, the case of non-compact M.

Suppose M is compact; then we can assume that J is finite. Repeated application of (11.6) leads to a situation in which M is obtained in the manner of (11.4) from two charts. It remains to show that the diffeomorphism type of M is then uniquely determined. We identify only one of the pairs of subintervals and apply (11.6). Then we see that M is, up to diffeomorphism, obtained from $]0,3[$ by identifying $]0,1[$ with $]2,3[$ under an increasing diffeomorphism ω. We show that the result is diffeomorphic to the one obtained from the standard case $\omega(x) = x+2$. For this purpose one uses again the method of (11.6). □

(2.11.7) Theorem. *A connected 1-manifold with non-empty boundary is diffeomorphic to $[0,1]$ or to $[0,1[$.*

2.12 Homotopy Spheres

A closed (smooth) n-manifold M is called a ***homotopy sphere***, if it is homotopy equivalent to the sphere S^n. In the case that M is not diffeomorphic to S^n, the manifold is called an ***exotic sphere***. The theorem of Hurewicz and Whitehead imply that for $n > 1$ a closed manifold is a homotopy sphere if and only if it is simply connected and $H*(M;\mathbb{Z})$ is isomorphic to $H_*(S^n;\mathbb{Z})$. From $H_n(M) \cong H_n(S^n) \cong \mathbb{Z}$ we see that a homotopy sphere is orientable.

(2.12.1) Proposition. *Let M be a homotopy n-sphere. Then $M \smallsetminus p$ is contractible. And also $M \smallsetminus U$, if $\overline{U}$ is homeomorphic to D^n.*

Proof. □

(2.12.2) Example. Twisted spheres Let $f\colon S^{n-1} \to S^{n-1}$ be a diffeomorphism. We use it in order to obtain from $D^n + D^n$ an n-manifold $M(f) = D^n \cup_f D^n$. It is homeomorphic to S^n. ◇

(2.12.3) Proposition. *If $f, g\colon S^{n-1} \to S^{n-1}$ are diffeotopic, then $M(f)$ and $M(g)$ are diffeomorphic. The connected sum $M(f)\#M(g)$ is oriented diffeomorphic to $M(fg)$.*

Proof. □

(2.12.4) Proposition. *The connected sum of two homotopy spheres is again a homotopy sphere.*

Proof. □

Let Θ_n be the set of oriented diffeomorphism classes of homotopy n-spheres. The connected sum induces on Θ_n an associative and commutative composition law with neutral element S^n.

(2.12.5) Proposition. *Let M be a homotopy n-sphere. Then $M\#(-M)$ is the boundary of a contractible orientable compact manifold.*

Proof. □

(2.12.6) Example. Let $a = (a_0, a_1, \ldots, a_n)$, $a_i \geq 2$ be integers. Let a_0 and a_1 be co-prime to the remaining integers a_j. Then the Brieskorn manifold $B(a)$ is a $\mathbb{Z}$-homology sphere. In the case that $n \geq 3$, $B(a)$ is simply connected and hence a homotopy sphere. ◇

An h-***cobordism*** $(B; M_0, M_1)$ between closed manifolds M_i is a bordism B between them such that the inclusions $M_i \subset B$ are homotopy equivalences.

(2.12.7) Theorem (h-cobordism theorem). *Let $n \geq 5$. Let B be an h-cobordism between n-manifolds M_i. Then there exists a diffeomorphism $B \to M_0 \times [0,1]$ which is on M_0 the identity. In particular M_0 and M_1 are diffeomorphic.*

(2.12.8) Proposition. *Let $n \geq 5$. Suppose the simply connected closed n-manifold M is the boundary of a compact contractible manifold W. Then M is diffeomorphic to S^n.*

(2.12.9) Corollary. *For $n \geq 5$ the element $[-M]$ is an inverse to $[M]$ in Θ. Thus we have the group of homotopy spheres Θ_n.*

(2.12.10) Proposition. *For $n \geq 6$ each homotopy sphere is a twisted sphere and in particular homeomorphic to a sphere.*

Index